Proposition 1

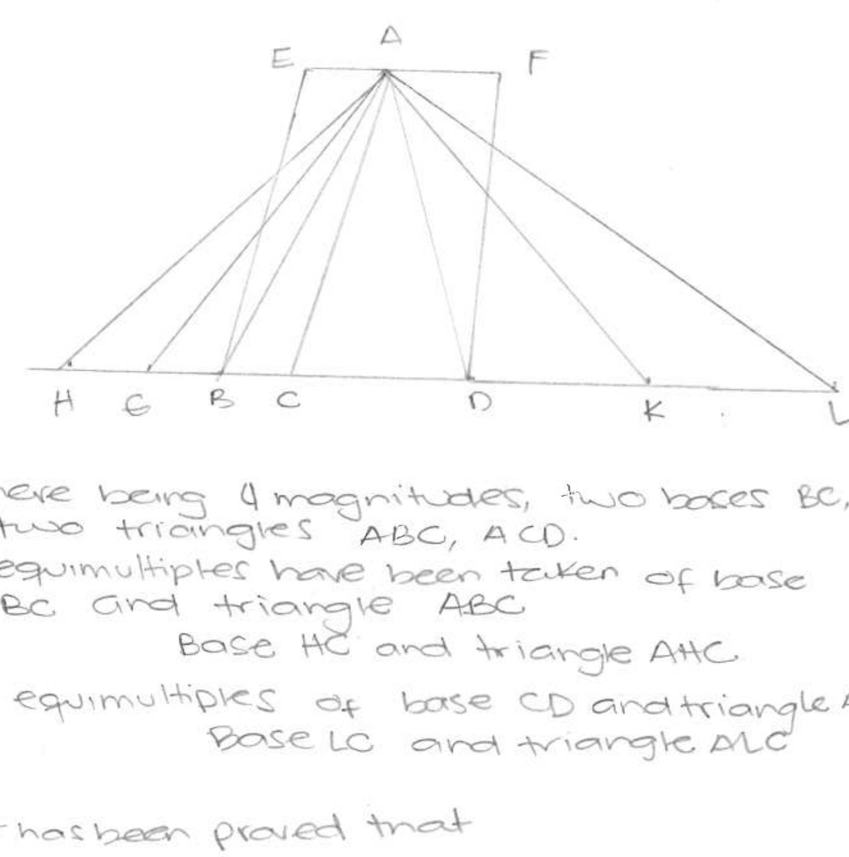
Triangles and parallelograms which are under the same height are to one another as their bases.

given:

△ABC △ACD
▱EC ▱CF
under the same height

required:

As the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and ▱EC is to ▱CF.



$BD \to H, L$

$\overline{BG}, \overline{GH}$ = base BC

$\overline{DK}, \overline{KL}$ = base CD

$\overline{AG}, \overline{AH}, \overline{AK}, \overline{AL}$

$CB = BG = GH$

$\triangle ABC = \triangle AGB = \triangle AHG$ [I.38]

whatever multiple the base HC is of base BC, that multiple is also the triangle AHC of triangle ABC.

For the same reason, whatever multiple base LC is of base CD, that multiple is the triangle ALC of the triangle ACD

If Base HC = base CL
$\triangle AHC = \triangle ACL$ [I.38]

If base HC is in excess of base CL
△AHC is in excess of △ACL
If less, less

There being 4 magnitudes, two bases BC, CD two triangles ABC, ACD.
equimultiples have been taken of base BC and triangle ABC
Base HC and triangle AHC
equimultiples of base CD and triangle ADC
Base LC and triangle ALC

It has been proved that
If base HC is in excess of base CL
Triangle AHC is in excess of △ALC

As base BC is to base CD
triangle ABC is to triangle ACD
[V.def.5]

→ Parallelogram EC is double △ABC [I.41]
▱FC is double △ACD
while parts have the same ratio as their multiple [V.15]

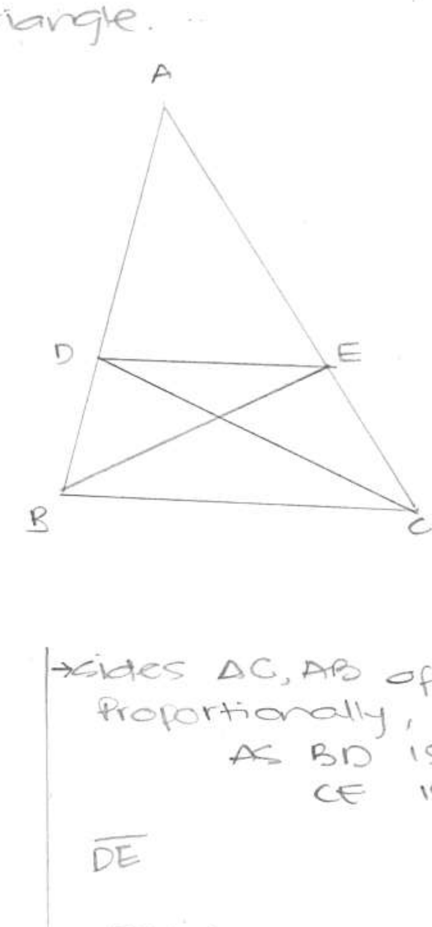
As △ABC is to △ACD
▱EC is to ▱FC

→ As base BC is to CD
△ABC is to △ACD
▱EC is to ▱CF [V.11]

Q.E.D.

Proposition 2

If a straight line be drawn parallel to one of the sides of the triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.



given

$\triangle ABC$

$\overline{DE} \parallel \overline{BC}$

required

AS BD is to DA
CE is to EA.

$\overline{BE}, \overline{CD}$

$\triangle BDE = \triangle CDE$
same base DE
same $\parallel$ DE, BC [I.38]

$\triangle ADE$ is another Area

Equals have the same ratio to the same [V.7]

AS $\triangle BDE$ is to $\triangle ADE$
$\triangle CDE$ is to $\triangle ADE$

AS $\triangle BDE$ is to $\triangle ADE$
BD is to AD
for, being under the same height, the perpendicular drawn from E to AB, they are to one another as their bases [VI.1]

For the same reason,
AS $\triangle CDE$ is to $\triangle ADE$
CE is to AE

AS BD is to AD
CE is to EA [V.11]

→ sides AC, AB of $\triangle ABC$ be cut proportionally,
AS BD is to DA
CE is to EA

$\overline{DE}$

→ DE $\parallel$ BC

with the same construction,
AS BD is to DA
CE is to EA

AS BD is to DA
$\triangle BDE$ is to $\triangle ADE$
AS CE is to EA
$\triangle CDE$ is to $\triangle ADE$ [VI.1]

AS $\triangle BDE$ is to $\triangle ADE$
$\triangle CDE$ is to $\triangle ADE$ [V.11]

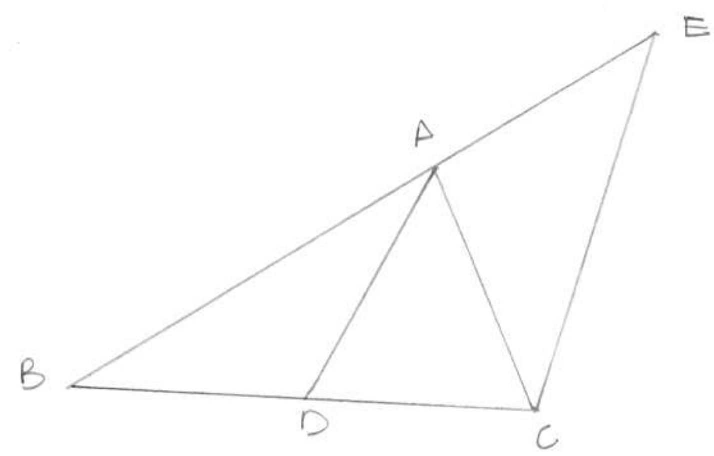
$\triangle BDE$, $\triangle CDE$ have the same ratio to $\triangle ADE$

$\triangle BDE = \triangle CDE$ [V,9]
same base DE
equal triangles which are on the same base are in the same parallels [I.39]

DE $\parallel$ BC Q.E.D.

Proposition 3

If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle; and, if the segments of the base have the same ratio as the remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.



Given.

$\triangle ABC$

$\measuredangle BAC$ bisected by $\overline{AD}$

Required

AS BD IS to CD
BA IS to AC

CE → C
CE ∥ DA

BA → E

AC falls on ∥ AD, EC
$\measuredangle ACE = \measuredangle CAD$;

→ hypothesis
$\measuredangle CAD = \measuredangle BAD$
$\measuredangle BAD = \measuredangle ACE$

BAE falls on ∥ AD, EC
$\measuredangle BAD = \measuredangle AEC$ [I.29]

$\measuredangle ACE = \measuredangle BAD$
$\measuredangle ACE = \measuredangle AEC$
AE = AC [I.6]

AD ∥ EC
AS BD IS to DE
BA IS to AE

AE = AC [VI.2]

→ let BA IS to AC
as BD to DC

$\overline{AD}$

→ $\measuredangle BAC$ has been bisected by $\overline{AD}$

with the same construction,
AS BD is to DC
BA IS to AC

AS BD IS to DC
BA IS to AE
AD ∥ EC [VI.2]

AS BA IS to AC
BA IS to AE [V.11]

AC = AE [V.9]
$\measuredangle AEC = \measuredangle ACE$ [I.5]

$\measuredangle AEC = \measuredangle BAD$ [I.29]
$\measuredangle ACE = \measuredangle CAD$ [I.29]

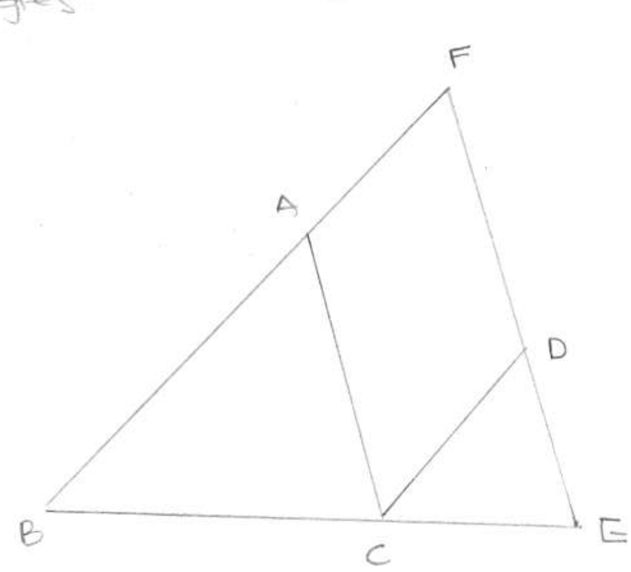
$\measuredangle BAD = \measuredangle CAD$

$\measuredangle BAC$ is besected by $\overline{AD}$

Q. E. D.

Proposition 4

In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles



given

△ ABC △ ADE are equiangular
∡ABC = ∡DCE
∡BAC = ∡CDE
∡ACB = ∡CED

required

In △ABC, △ADE the sides about the ∡ are proportional, and those are corresponding sides which subtend the equal ∡

$\overline{BC} \rightarrow \overline{CE}$

∡ABC, ∡ACB < 2∟ [I.17]
∡ACB = ∡DEC

∡ABC, ∡DEC < 2∟
BA, ED produced will meet [I. Post.5]

$\overline{BA}, \overline{ED} \rightarrow F$

∡DCE = ∡ABC
BF ∥ CD [I.28]

∡ACB = ∡DEC
AC ∥ EF [I.28]

▱FACD
FA = CD
AC = FD [I.34]

AC ∥ FE
one side of △FBE
AS BA is to AF
BC is to CE [VI.2]

AF = CD
AS BA is to CD
BC is to CE

alternately,
AS AB is to BC
CD is to CE [V.16]

CD ∥ BF
AS BC is to CE
AC is to DE
alternately,
AS BC is to AC
CE is to DE [V.16]

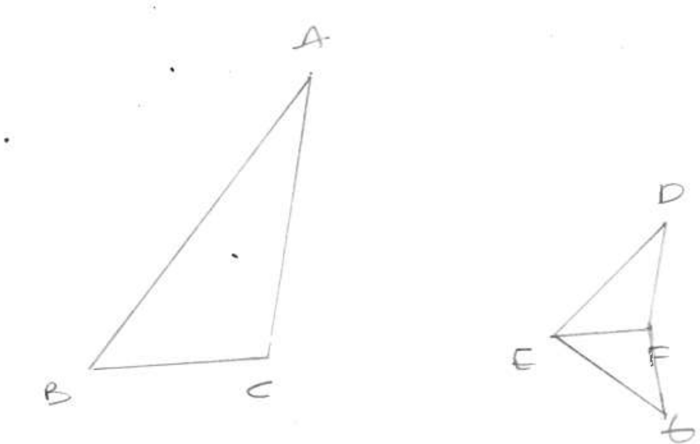
It was proved that:
AS AB is to BC
DE is to CE
AS BC is to CA
CE is to ED.

ex aequali:
AS BA is to AC
CD is to DE [V.22]

Q.E.D.

Proposition 5

If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.



given:

$\triangle ABC$, $\triangle DEF$ have their sides proportional

AS AB is to BC
DE is to EF

AS BC is to CA
EF is to FD

AS BA is to AC
ED is to DF

Required:

$\triangle ABC$ is equiangular with $\triangle DEF$

$\angle ABC = \angle DEF$

$\angle BCA = \angle EFD$

$\angle BAC = \angle EDF$

on $\overline{EF}$

$\angle FEG = \angle ABC$

$\angle EFG = \angle ACB$ [I.23]

remaining $\angle A$ = remaining $\angle G$ [I.32]

$\triangle ABC$ equiangular to $\triangle GEF$

In $\triangle ABC$, $\triangle GEF$ the sides about the equal $\angle$ are proportional; and those corresponding sides which subtend the equal $\angle$'s [VI.4]

AS AB is to BC
GE is to EF

AS AB is to BC
DE is to EF

AS DE is to EF
GE is to EF [V.11]

DE, EG have the same ratio to EF
DE = EG [V.9]

For the same reason,
DF = GF

DE = EG
EF common
DE, EF = EG, EF
base DF = base FG
$\angle DEF = \angle GEF$ [I.8]
$\triangle DEF = \triangle GEF$

The remaining $\angle$ = remaining $\angle$; those which the $\angle$ subtend [I.4]

$\angle DFE = \angle GFE$
$\angle EDF = \angle EGF$

$\angle FED = \angle GEF$
$\angle GEF = \angle ABC$

$\angle ABC = \angle FED$

For the same reason,

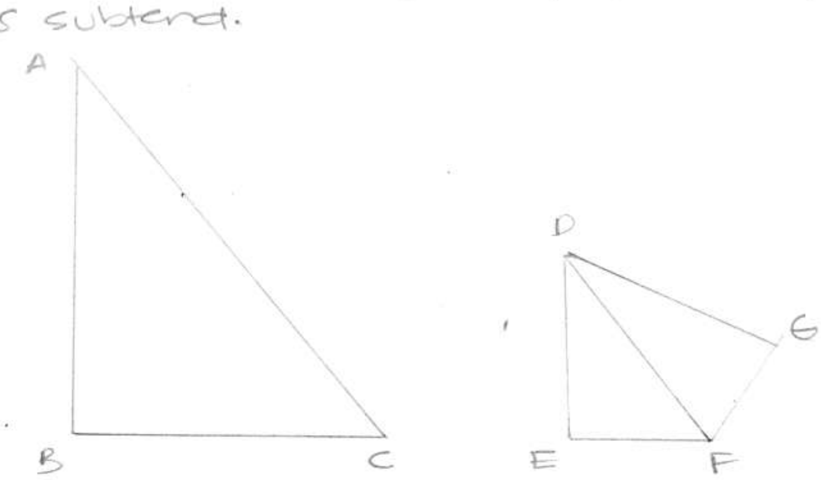
$\angle ACB = \angle DEF$
$\angle A = \angle D$

$\triangle ABC$ is equiangular with $\triangle DEF$.

Q.E.D.

Proposition 6

If two triangles have one angle equal to one angle and the side about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.



Given

$\triangle ABC$ $\triangle DEF$

$\measuredangle BAC = \measuredangle EDF$

sides about the equal $\measuredangle$ proportional

Required

$\triangle ABC$ is equiangular to $\triangle DEF$

$\measuredangle ABC = \measuredangle DEF$

$\measuredangle ACB = \measuredangle DFE$

on $\overline{DF}$

$\measuredangle FDG = \measuredangle BAC$ or $\measuredangle EDF$

$\measuredangle DFG = \measuredangle ACB$ [1.23]

$\measuredangle B = \measuredangle G$ [1.32]

$\triangle ABC$ is equiangular $\triangle DGF$

Proportionally,

As BA is to AC
GD is to DF [VI.4]

As BA is to AC
ED is to DF

As ED is to DF
GD is to DF [V.11]

ED = DG [V.9]

DF common

ED, DF = DG, DF

$\measuredangle EDF = \measuredangle GDF$

$\triangle DEF = \triangle DGF$

remaining $\measuredangle$ = remaining $\measuredangle$;
those which = $\measuredangle$ subtend [1,4]

$\measuredangle DFE = \measuredangle DFG$

$\measuredangle DEF = \measuredangle DGF$

$\measuredangle DFG = \measuredangle ACB$

$\measuredangle ACB = \measuredangle DFE$

$\measuredangle BAC = \measuredangle EDF$

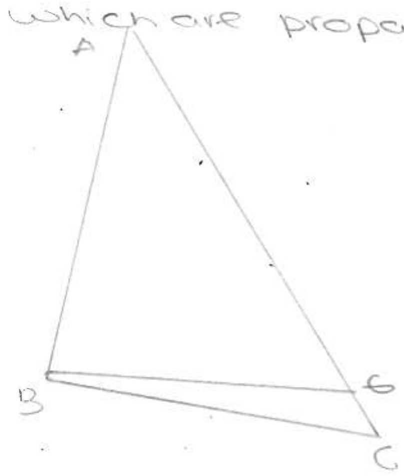
$\measuredangle B = \measuredangle E$ [1.32]

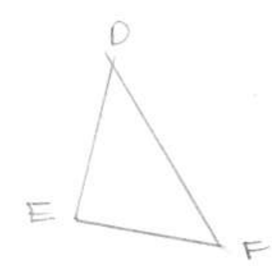
$\triangle ABC$ is equiangular to $\triangle DEF$

Q.E.D.

Proposition 7

If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional





given:

$\triangle ABC, \triangle DEF$

$\measuredangle BAC = \measuredangle EDF$

The sides about $\measuredangle ABC, \measuredangle DEF$ proportional,

As AB is to BC
DE is to EF

The remaining $\measuredangle$ at C, F $< \llcorner$

required:

$\triangle ABC$ equiangular $\triangle DEF$

$\measuredangle ABC = \measuredangle DEF$

$\measuredangle C = \measuredangle F$

→ If $\measuredangle ABC$ is unequal $\measuredangle DEF$ one is greater

$\measuredangle ABC > \measuredangle DEF$

on AB
$\measuredangle ABG = \measuredangle DEF$ [I.23]

$\measuredangle A = \measuredangle D$

$\measuredangle ABG = \measuredangle DEF$
$\measuredangle AGB = \measuredangle DFE$ [I.32]

$\triangle ABG$ equiangular $\triangle DEF$

As AB is to BG
DE is to EF [VI.4]

as DE is to EF
AB is to BC

AB has the same ratio to BC, BG [V.11]
BC = BG [V.9]
$\measuredangle C = \measuredangle BGC$ [I.5]

$\measuredangle C < \llcorner$
$\measuredangle BGC < \llcorner$
$\measuredangle AGB > \llcorner$ [I.13]

$\measuredangle AGB = \measuredangle F$
$F > \llcorner$

But it is less than a right $\measuredangle$:
ABSURD

$\measuredangle ABC = \measuredangle DEF$

$\measuredangle A = \measuredangle D$
$\measuredangle C = \measuredangle F$ [I.32]

→ let $\measuredangle C$, $\measuredangle F$ be supposed not less than ∟

$\triangle ABC$ equiangular $\triangle DEF$

with the same construction,

$BC = BG$

$\measuredangle C = \measuredangle BGC$ [1.5]

$\measuredangle C$ is not less than ∟

$\measuredangle BGC$ is not less than ∟

In $\triangle BGC$

two $\measuredangle$'s are not less than two right $\measuredangle$: impossible [1.17]

$\measuredangle ABC = \measuredangle DEF$

$\measuredangle A = \measuredangle D$

$\measuredangle C = \measuredangle F$ [1.32]

$\triangle ABC$ equiangular $\triangle DEF$

Q.E.D

Proposition 8

If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.

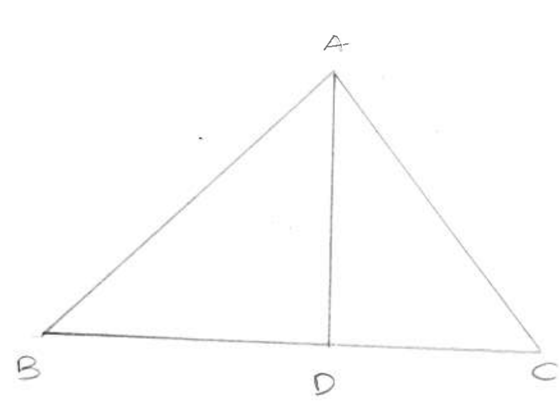
Given.

ΔABC right-angled
$\measuredangle BAC = ⦜$

$\overline{AD}$ perpendicular to BC

Required:

ΔABD, ΔADC is similar to ΔABC and to one another.



$\measuredangle BAC = \measuredangle ADB$
each is ⦜

$\measuredangle B$ is common to ΔABC, ΔABD
$\measuredangle ACB = \measuredangle BAD$ [I.32]
ΔABC equiangular ΔABD

As,
BC subtends the ⦜ in ΔABC
BA subtends the ⦜ in ΔABD

As
AB subtends the equal $\measuredangle BAD$ in ΔABD so is also AC to AD which subtends the $\measuredangle B$ common to both Δ [VI.4]

ΔABC equiangular to ΔABD
sides about equal $\measuredangle$ proportional

ΔABC similar to ΔABD [VI.def.1]

Similarly,
ΔABC similar to ΔADC

ΔABC, $\measuredangle ADC$ similar to whole ΔABC

$\rightarrow \Delta ABD$ similar to $\measuredangle ADC$.

$\measuredangle BDA = \measuredangle ADC$
each is ⦜

$\measuredangle BAD = \measuredangle C'$
$\measuredangle B = \measuredangle DAC$ [I.32]

ΔABD equiangular to ΔADC

As BD subtends $\measuredangle BAD$ in ΔABD
DA subtends $\measuredangle C$ in ΔADC
equal to $\measuredangle BAD$.
so is AD which subtends $\measuredangle B$ in ΔABD to DC which subtends $\measuredangle DAC$ in ΔADC
equal to $\measuredangle B$
also is BA to AC, the sides subtending ⦜ [VI.4]

ΔABD similar to ΔAbc
[VI.def.1]

Q.E.D

PORISM:

From this it is clear that, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the straight line so drawn is a mean proportional between the segments of the base.

Proposition 9

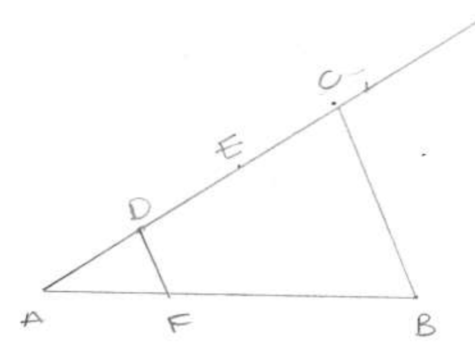
From a given straight line to cut off a prescribed part.

given

$\overline{AB}$

required.

to cut off from
AB a prescribed
Part.



let the third part be
Prescribed

$\overline{AC} \rightarrow A$
containing any angle with AB
∘D at random on AC
$\overline{DE}, \overline{EC} = \overline{AD}$ [I.3]

$\overline{BC}$
$\overline{DF} \parallel \overline{BC}$ [I.31]

FD ∥ BC
one of the sides of △ABC
Proportionally,
As CD is to DA
BF is to FA [VI.2]

CD is double DA
BF is double FA
BA is triple FA

given $\overline{AB}$ the prescribed third part AF
has been cut off

Q.E.D.

Proposition 10

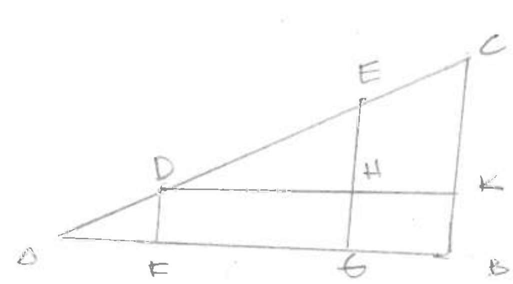
To cut a given uncut straight line similarly to a given cut straight line.

given

$\overline{AB}$ uncut straight line
$\overline{AC}$ cut at D, E.

required

cut $\overline{AB}$ similarly to $\overline{AC}$



$\overline{CB}$

DF, EG ∥ BC

DHK ∥ AB [I.31]

▱ FH
▱ HB

DH = FG
HK = GB [I.34]

HE ∥ KC
one of the sides of △DKC
Proportionally,
AS CE is to ED
HK is to HD [VI.2]

KH = GB
HD = FG

AS CE is to ED
GB is to FG

FD ∥ GE
one of the sides of △AGE
Proportionally,
AS ED is to DA
GF is to FA [VI.2]

AS CE is to ED
BG is to FG

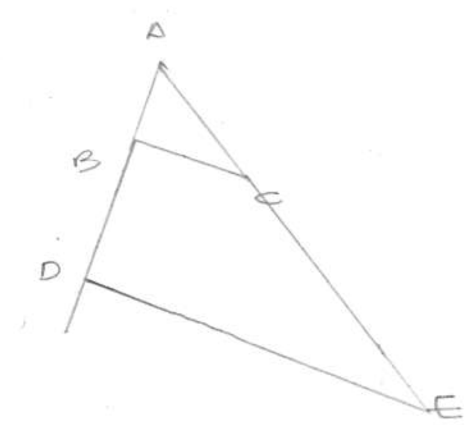
AS ED is to DA
GF is to FA

a given uncut line $\overline{AB}$ has been cut similarly to the given cut line $\overline{AC}$

Q.E.F.

Proposition 11

To two given lines to find a third proportional



given:
$\overline{BA}, \overline{AC}$

required
Third proportional
to BA, AC

$\overline{AB} \to D$
$\overline{AC} \to E$
$\overline{BD} = \overline{AC}$
$\overline{BC}$
$DE \parallel BC$ [I.31]

$BC \parallel DE$
one of the sides of $\triangle ADE$
Proportionally,
AS AB is to BD
AC is to CE [VI.2]

$BD = AC$

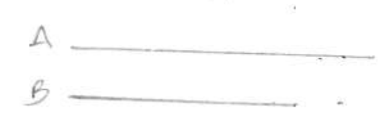
AS AB is to AC
AC is to CE

given $\overline{AB}, \overline{AC}$ a third proportional to them, $\overline{CE}$
has been found.

Q.E.F.

Proposition 12

To three given straight lines to find a fourth proportional

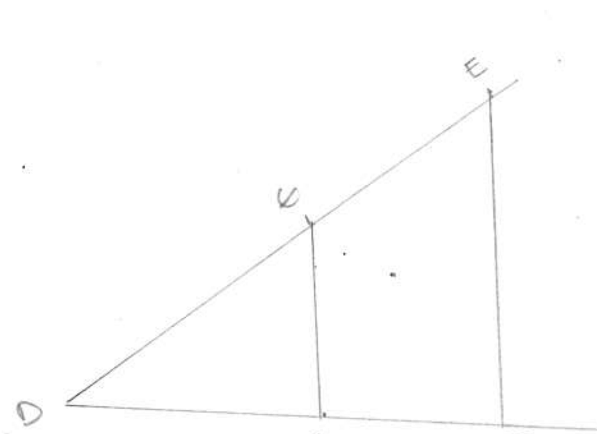


given:

$\overline{A}, \overline{B}, \overline{C}$

required:

find a fourth
Proportional to
A, B, C.



$\overline{DE}, \overline{EF}$

$\overline{DG} = A$
$\overline{GE} = B$
$\overline{DH} = C$

$\overline{GH}$

EF ∥ GH [I.31]

GH ∥ EF
one of the sides of △ DEF
As DG is to GE
DH is to HF [VI.2]

DG = A
GE = B
DH = C

As A is to B
C is to HF

given $\overline{A}, \overline{B}, \overline{C}$ a fourth proportional HF has been found.

Q.E.F.

Proposition 13

two given straight lines to find a mean proportional

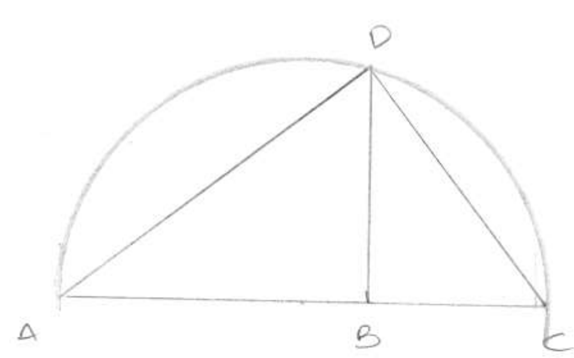
given:

$\overline{AB}$

$\overline{BC}$

required.

to find a mean
Proportional to $\overline{AB}$, $\overline{BC}$



Semicircle described on $\overline{AC}$

$\overline{BD}$ at $\perp$ to AC

$\overline{AD}$, $\overline{DC}$

$\measuredangle ADC$ is the angle in a semicircle

$\measuredangle ADC = \perp$ [III.31]

In the right-angled triangle ADC,
DB has been drawn from the right
angle perpendicular to the base

DB is a mean proportional between
the segments of the base. AB, BC

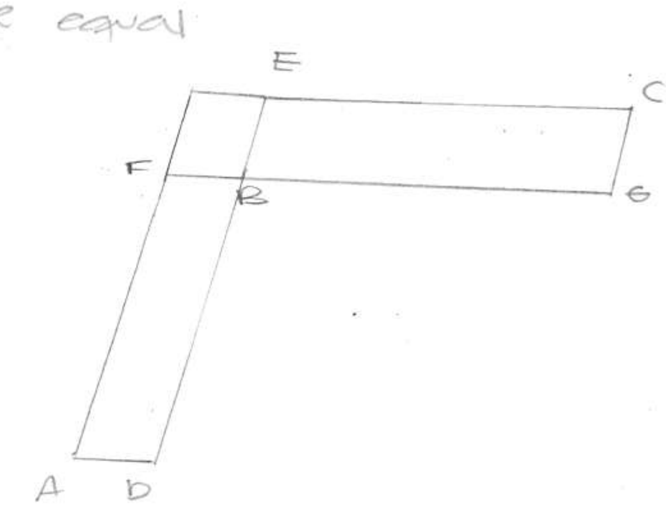
[VI. 8, Por]

given $\overline{AB}$, $\overline{BC}$ a mean proportional
DB has been found.

Q.E.F.

Proposition 14

In equal and equiangular parallelograms. the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal



Given:

▱AB, ▱BC
equal and equiangular,
angles at B equal
$\overline{DB}, \overline{BE}$ in a straight line
FB, BG in a straight line [I.14]

Required:

In AB, BC the sides about the equal angles are reciprocally proportional.

AS DB is to BE
GB is to BF

Complete ▱FE

▱AB = ▱BC
FE is in another area

AS AB is to FE
BC is to FE [V.7]

AS AB is to FE
DB is to BE [VI.1]

AS BC is to FE
BG is to BF [VI.1]

AS DB is to BE
GB is to BF [V.11]

▱AB, ▱BC The sides about the equal angles are reciprocally proportional.

→ let GB be to BF
as DB to BE
▱AB = ▱BC

AS DB is to BE
GB is to BF

AS DB is to BE
▱AB is to ▱FE [VI.1]

AS GB is to BF
▱BC is to ▱FE [VI.1]

AS AB is to FE
BC is to FE [V.11]

▱AB = ▱BC [V.9]

Q.E.D

Proposition 15

In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.

given.

$\Delta ABC = \Delta ADE$

$\measuredangle BAC = \measuredangle DAE$

required.

The sides about the equal angles are reciprocally proportional

AS CA is to AD
EA is to AB

CA in straight line with AD.
EA in straight line with AB [I,14]

$\overline{BD}$

$\Delta ABC = \Delta ADE$
BAD is in another area

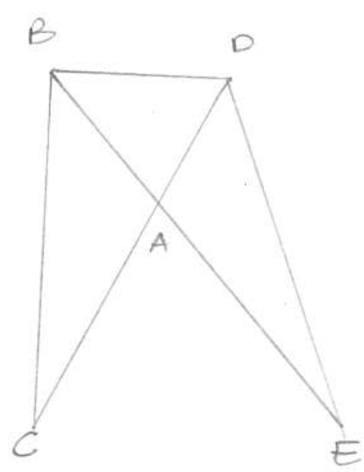
AS ΔABC is to ΔBAD
ΔADE is to ΔADE [V.7]

AS CAB is to BAD
CA is to AD [VI.1]

AS EAD is to BAD
EA is to AB [V.1]

AS CA is to AD
EA is to AB [V.11]

in $\Delta ABC, \Delta ADE$ sides about the equal angles are reciprocally proportional



given:
~~sides~~ of ΔABC, ΔADE be reciprocally proportional
AS EA to AB
CA is to AD

required

$\Delta ABC = \Delta ADE$

$\overline{BD}$

AS CA is to AD
EA is to AB

AS CA is to AD
ΔABC is to BAD

AS EA is to AB
ΔEAD is to ΔBAD [VI.1]

AS ΔABC is to ΔBAD
ΔEAD is to ΔBAD [V.11]

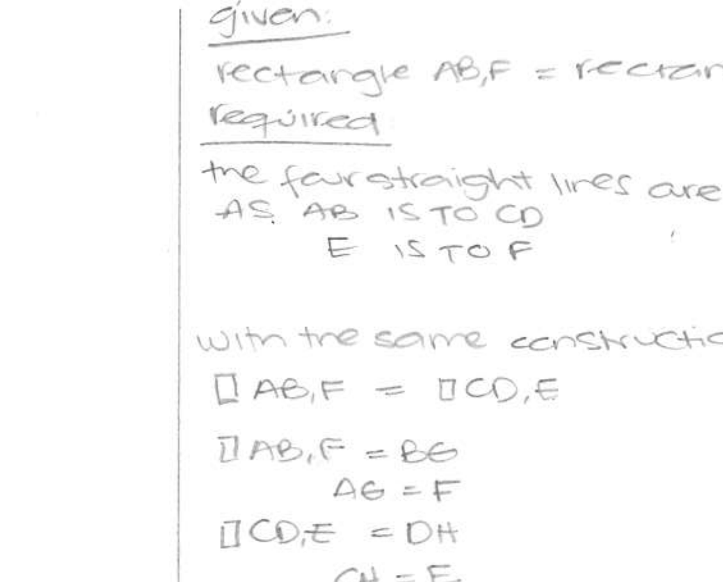
$\Delta ABC, \Delta EAD$ have the same ratio to ΔBAD

$\Delta ABC = \Delta EAD$ [V.9]

Q.E.D.

Proposition 16

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional.



given:

$\overline{AB}, \overline{CD}, \overline{E}, \overline{F}$ be proportional

AS AB is to CD
E is to F

required:

rectangle contained by AB,F = rectangle contained by CD,E

$\overline{AG}, \overline{CH}$ from A,C at ⊥ to AB, CD

$\overline{AG} = F$

$\overline{CH} = E$

▱BG, ▱DH

AS AB is to CD
E is to F

AG = F, CH = E

AS AB is to CD
CH is to AG

in ▱BG, ▱DH the sides about the equal angles are reciprocally proportional

equiangular parallelograms in which sides about the equal angles are reciprocally proportional are equal [VI.14]

▱BG = ▱DH

BG is the rectangle AB,F
for AG = F

DH is the rectangle CD,E
for CH = E

rectangle AB,F = rectangle CD,E.

given:

rectangle AB,F = rectangle CD,E

required

the four straight lines are proportional
AS AB is to CD
E is to F

with the same construction,

▭AB,F = ▭CD,E

▭AB,F = BG
AG = F

▭CD,E = DH
CH = E

BG = DH

they are equiangular

in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional

AS AB is to CD
CH is to AG

CH = E
AG = F

AS AB is to CD
E is to F

Q.E.D

Proposition 17

If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines will be proportional.

given.

$\overline{A}, \overline{B}, C$ be proportional

AS A IS to B
B IS to C

Required:

rectangle contained by A, C = square on B

$\overline{D}$

D = B

AS A IS TO B
B IS TO C

B = D

AS A IS to B
D IS to C

If four straight lines be Proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means
[VI.16]

rectangle A, C = rectangle B, D

rectangle B, D = square B
B = D

rectangle A, C = square B

given.

rectangle A, C = square B

required:

AS A is to B
B is to C

with the same construction,
rectangle A, C = square B

square B = rectangle B, D
B = D

rectangle A, C = rectangle B, D

If the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportional
[VI.16]

AS A is to B
D is to C

B = D

AS A is to B
B is to C

Q.E.D.

Proposition 18

On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure

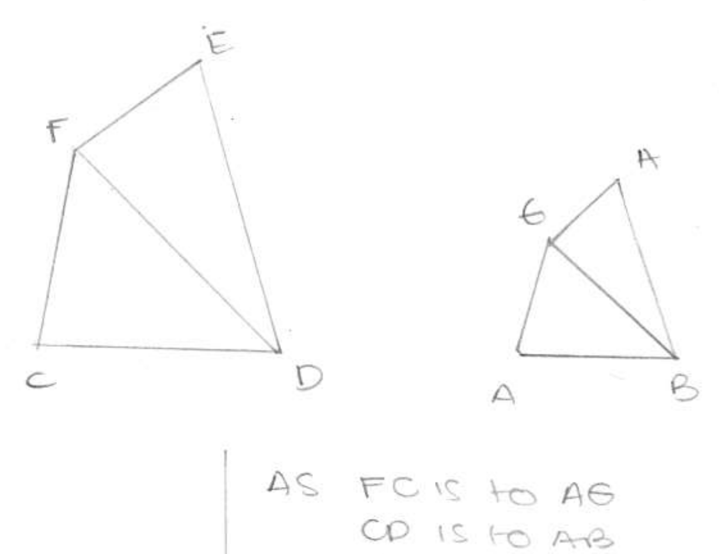
given

$\overline{AB}$

rectilineal figure CE

required:

describe on $\overline{AB}$ a rectilineal figure similar and similarly situated to CE



$\overline{DF}$

on $\overline{AB}$

$\angle GAB = \angle C$

$\angle ABG = \angle CDF$ [I.23]

$\angle CFD = \angle AGB$ [I.32]

$\triangle FCD$ equiangular $\triangle GAB$

AS FD is to GB

FC is to GA

CD is to AB

on $\overline{BG}$

$\angle BGH = \angle DFE$

$\angle GBH = \angle FDE$ [I.23]

$\angle E = \angle H$ [I.32]

$\triangle FDE$ equiangular $\triangle GBH$

AS FD is to GB

FE is to GH

ED is to BH [VI.4]

But it was also proven,

AS FD is to GB

FC is to GA

CD is to AB

AS FC is to AG

CD is to AB

FE is to GH

ED is to HB

$\angle CFD = \angle AGB$

$\angle DFE = \angle BGH$

$\angle CFE = \angle AGH$

For the same reason

$\angle CDE = \angle ABH$

$\angle C = \angle A$

$\angle E = \angle H$

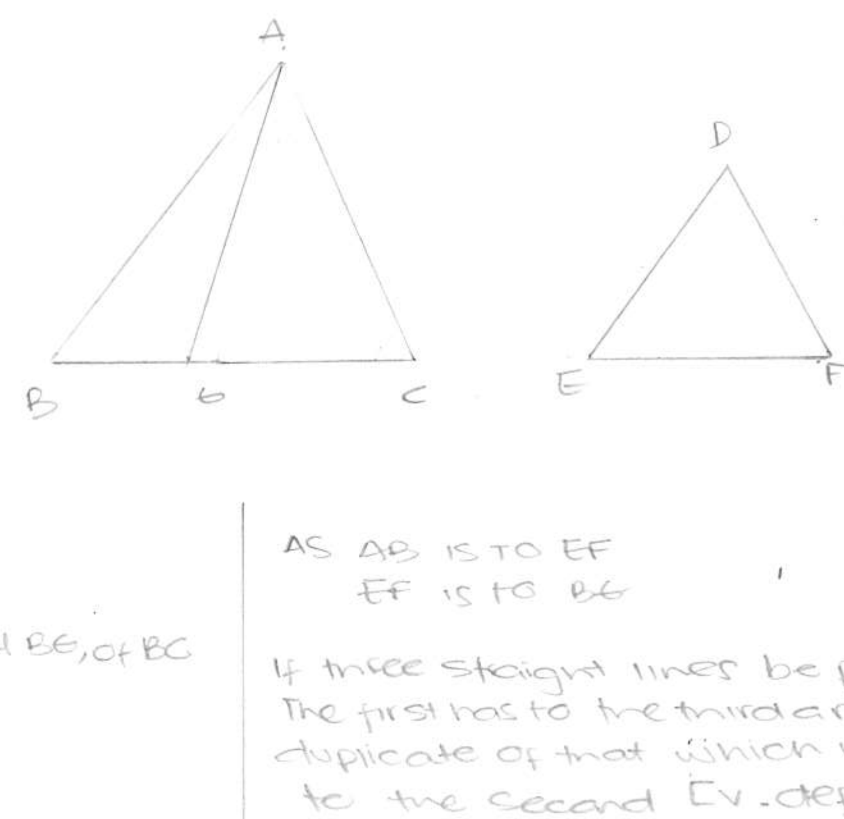
AH is equiangular to CE and the sides about their equal angles are proportional;

rectilineal figure AH is similar to rectilineal figure CE [VI. def. 1]

Q.E.D.

Proposition 19

Similar triangles are to one another in the duplicate ratio of the corresponding sides.



Given.

$\triangle ABC$, $\triangle DEF$ be similar

$\angle B = \angle E$

As AB is to BC
  DE is to EF

BC corresponds to
  EF [V.def 11]

Required.

$\triangle ABC$ has to $\triangle DEF$ a ratio duplicate of that which BC has of EF

Take a third proportional BG, of BC
As BC is to EF
  EF is to BG

$\overline{AG}$

As AB is to BC
  DE is to EF

alternately,

As AB is to DE
  BC is to EF [V.16]

But,
As BC is to EF
  EF is to BG

As AB is to DE
  EF is to BG [V.11]

In $\triangle ABG$, $\triangle DEF$ the sides about the equal $\angle$ are proportional.

But those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal [VI.15]

$\triangle ABG = \triangle DEF$

As AB is to EF
  EF is to BG

If three straight lines be proportional, the first has to the third a ratio duplicate of that which it has to the second [V.def.9]

BC has to BG a ratio duplicate of that which CB has to EF

As CB is to BG
  $\triangle ABC$ is to $\triangle DEF$

triangle ABC has to $\triangle DEF$ a ratio duplicate, as that which BC has to EF

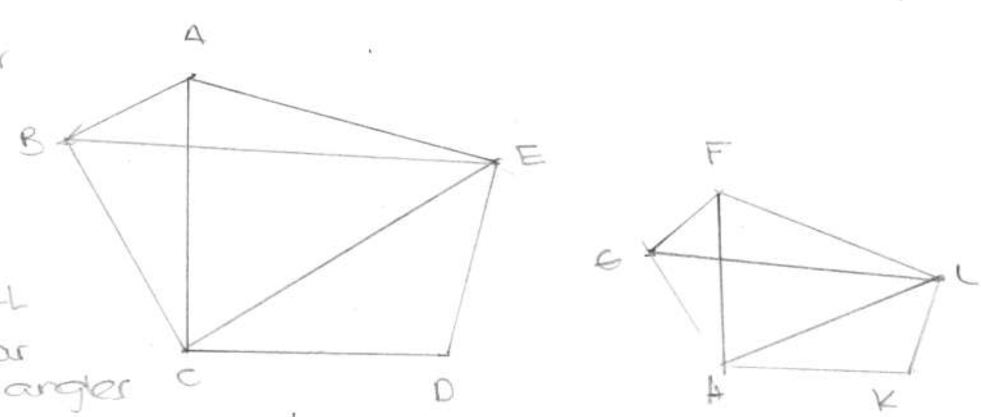
Q.E.D

Porism:

From this it is manifest that, if three straight lines be proportional, then, as the first is to the third, so is the figure described on the first to that which is similar and similarly described on the second.

Proposition 20

Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.



given:

ABCDE, FGKHL similar polygons.

AB correspond to FG

required:

Polygons ABCDE, FGHKL are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes; Polygon ABCDE has to Polygon FGHKL a ratio duplicate of that which AB has to FG.

$\overline{BE}$, $\overline{EC}$, $\overline{GL}$, $\overline{LH}$

Polygon ABCDE is similar to Polygon FGKHL

$\measuredangle BAE = \measuredangle GFL$

As BA is to EA
GF is to FL [VI. def. 1]

$\triangle ABE$, $\triangle FGL$ have one angle equal to one angle and the sides about the angle are proportional.

$\triangle ABE$ equiangular $\measuredangle FGL$ [VI. 6]
It is also similar [VI. 4 and Def. 1]
$\measuredangle ABE = \measuredangle FGL$

$\measuredangle ABC = \measuredangle FGH$
because of polygon similarity
$\measuredangle EBC = \measuredangle LGH$

Because of similarity,
As EB is to BA
LG is to GF

As AB is to BC
FG is to GH

∴ ex aequali,

As EB is to BC
LG is to GH [V. 22]

The sides about equal angles $\measuredangle EBC$, $\measuredangle LGH$ are proportional
$\triangle EBC$ equiangular $\triangle LGH$ [VI. 6]
$\triangle EBC$ similar $\triangle LGH$ [VI. 4 and def 1]

For the same reason,
$\triangle ECD = \triangle LKH$

The similar polygons ABCDE and FGHKL have been divided into similar triangles and triangles equal in multitude.

required:

→ They are also in the same ratio as the wholes, in such manner that the triangles are proportional ABE, EBC, ECD are antecedents FGL, LGH, LHK are consequents. Polygon ABCDE has to polygon FGHKL a ratio duplicate of that which the corresponding side has to the corresponding side, AB to FG

$\overline{AC}$, $\overline{FH}$

Because of the similarity of polygons,
$\measuredangle ABC = \measuredangle FGH$
As AB is to BC
FG is to GH
$\triangle ABC$ equiangular $\triangle FGH$ [VI.6]
$\measuredangle BAC = \measuredangle GFH$
$\measuredangle BCA = \measuredangle GHF$

$\measuredangle BAM = \measuredangle GFN$
$\measuredangle ABM = \measuredangle FGN$
$\measuredangle AMB = \measuredangle FNG$ [I.32]
$\triangle ABM$ equiangular $\triangle FGN$
Similarly,
$\triangle BMC$ equiangular $\triangle GNH$

Proportionally,
As AM is to MB
FN is to NG
As BM is to MC
GN is to NH

ex aequali,
As AM is to MC
FN is to NH

As AM is to MC
$\triangle ABM$ is to $\triangle MBC$
$\triangle AME$ is to $\triangle EMC$
They are to one another as their bases [VI.1]

as one of the antecedents is to one of the consequents, so are all antecedents to all consequents [V.12]

As $\triangle AMB$ is to $\triangle BMC$
$\triangle ABE$ is to $\triangle CBE$

As AMB is to BMC
AM is to MC
As AM is to MC
$\triangle ABE$ is to $\triangle EBC$

For the same reason,
As FN is to NH
$\triangle FGL$ is to $\triangle GLH$

As AM is to MC
FN is to NH → As $\triangle ABE$ is to $\triangle BEC$
$\triangle FGL$ is to $\triangle GLH$

alternately,
as $\triangle ABE$ is to $\triangle FGL$
$\triangle BEC$ is to $\triangle GHL$

Similarly,
If BD, GK were joined.
As $\triangle BEC$ is to $\triangle LGH$
$\triangle ECD$ is to $\triangle LHK$

As $\triangle ABE$ is to $\triangle FGL$
EBC is to LGH
ECD is to LHK.

↳ as one of the antecedents is to one of the consequents so are all of the antecedents to all consequents [V.12]
As $\triangle ABE$ is to $\triangle FGL$
Polygon ABCDE is to polygon FGHKL

$\triangle ABE$ has to $\triangle FGL$ a ratio duplicate of that which corresponding side AB has to side FG. For, similar triangles are in duplicate ratio of the corresponding side [VI.19]
Also,
Polygon ABCDE has to polygon FGHKL a ratio duplicate of that which corresponding side AB has to corresponding side FG

Q.E.D.

Porism:

similarly also it can be proved in the case of quadrilaterals that they are in duplicate ratio of the corresponding sides. And it was also proved in case of triangles; therefore also, generally, similar rectilineal figures are to one another in the duplicate ratio of the corresponding sides.

Proposition 21

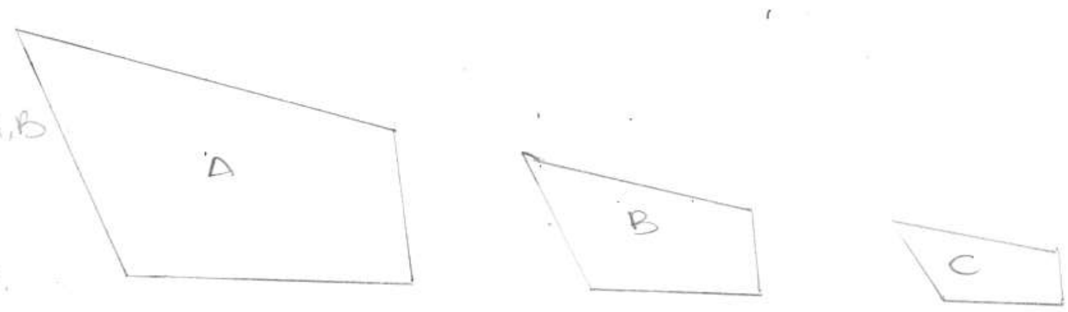
Figures which are similar to the same rectilineal figure are also similar to one another.

given

rectilineal figures A, B similar to C

required

A is similar to B.



A is similar to B,

It is equiangular with it and has the sides about the equal angles proportional [VI. def. 1]

B is similar to C

It is equiangular with it and has the sides about the equal angles proportional

∴ figures A, B are equiangular with C, and with C has the sides about the equal angles proportional;

A is similar to B

QED.

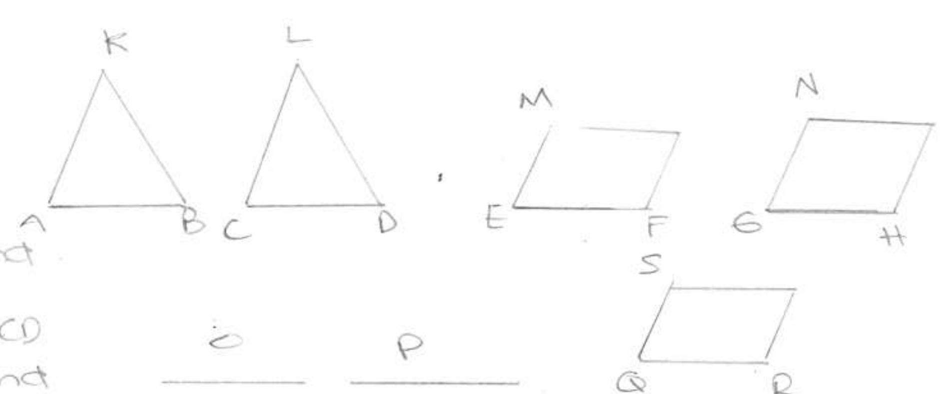
Proposition 22

If four straight lines be proportional, the rectilineal figures similar and similarly described upon them will also be proportional; and if the rectilineal figures similar and similarly described upon them be proportional, the straight line will themselves also be proportional.

given:

$\overline{AB}, \overline{CD}, \overline{EF}, \overline{GH}$

AS AB is to CD
EF is to GH



on AB, CD the similar and similarly situated rectilineal figure KAB, LCD
on EF, GH the similar and similarly situated rectilineal figures MF to NH

required:

AS KAB IS TO LCD
MF IS TO NH

let there be taken a third proportional O to AB, CD and a third proportional P to EF, GH [VI.11]

AS AB IS TO CD
EF is to GH

AS CD is to O
GH is to P

ex aequali,
AS AB is to O
EF is to P [V.22]

AS AB is to O
KAB is to LCD [VI.19, Por]

AS EF is to P
MF is to NH

AS KAB is to LCD
MF is to NH [V.11]

given:

let MF be to NH as KAB is to LCD;

required:

AS AB is to CD
EF is to GH

if EF is not to GH as AB to CD,
let EF be to QR as AB to CD. [VI.12]
and on QR let the rectilineal figure SR be described similar and similarly situated to either MF, NH.
[VI.18]

AS AB is to CD
EF is to QR
and on AB, CD the similar and similarly figures KAB, LCD
and on EF, QR the similar and similarly situated figures MF, SR

AS KAB IS TO LCD
MF IS TO SR

by hypothesis,
AS KAB IS TO LCD
MF is to NH $\rightarrow$ AS MF is to SR
MF is to NH [V.11]

MF has the same ratio to each of the figures NH, SR
NH = SR [V.9]

it is also similar and similarly situated to it;
GH = QR

AS AB is to CD
EF is to QR $\rightarrow$ QR = GH $\rightarrow$ AS AB is to CD
EF is to GH
Q.E.D

Proposition 23

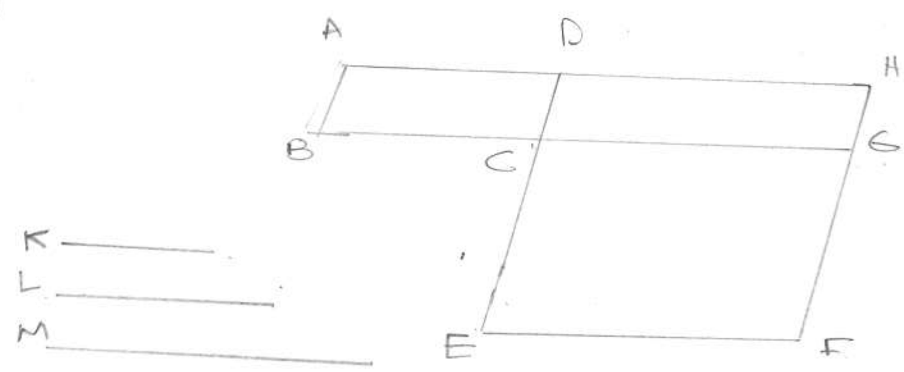
Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides

given:

AC, CF be equiangular parallelograms having The ∡BCD = ∡ECG

required:

▱AC has to ▱CF the ratio compounded of the ratios of the sides.



BC is in straight line with CE;
DC is in straight line with CE.

Complete ▱DG

K

AS BC is to CG
K is to L

AS DC is to CE
L is to M [VI.12]

ratios of K, L and of L, M are the same ratios of the sides BC to CG, DC to CE.

ratio K to M is compounded of the ratio K to L and L to M.

K has to M the same ratio compounded of the ratio of the sides.

AS BC is to CG
▱AC is to ▱CH [VI.1]

AS BC is to CE
K is to L

AS K is to L
AC is to CH [V.11]

AS DC is to CE
▱CH is to ▱CF [VI.1]

AS DC is to CE
L is to M

AS L is to M
▱CH is to ▱CF [V.11]

AS K is to L
▱AC is to ▱CH

AS L is to M
▱CH is to ▱CF

ex aequali,
as K is to M
▱AC is to ▱CF

K has to M the ratio compounded of the ratios of the sides.

AC has to CF the ratio compounded of the ratios of the sides.

QED.

Proposition 24

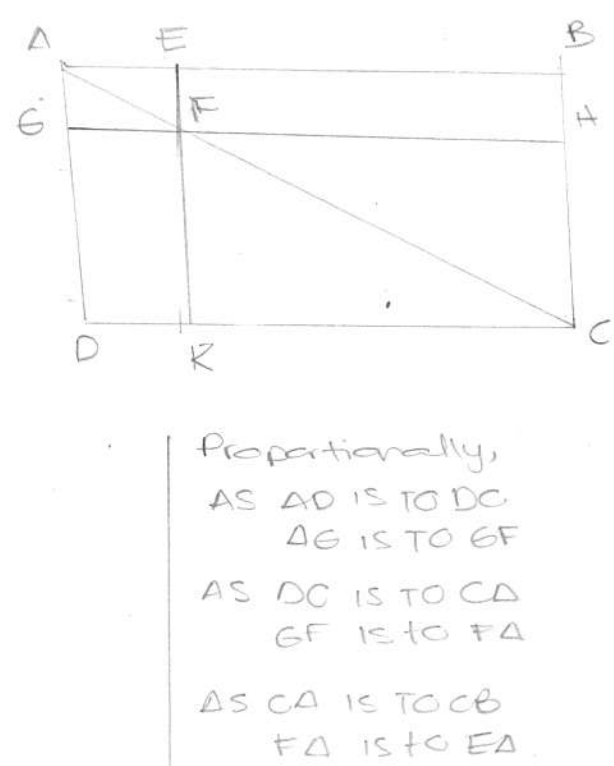
In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another.

Given:

▱ABCD
AC its diameter
EG, HK parallelograms about AC

Required:

▱EG, HK is similar to ▱ABCD and to the other



EF ∥ BC
one of the sides of △ABC
As BE is to EA
CF is to FA [VI.2]

FG ∥ CD
one of the sides of △ACD
As CF is to FA
DG is to GA [VI.2]

As CF is to FA
BE is to EA
As BE is to EA
DG is to GA

Componendo,
As BA is to EA
DA is to AG [V.18]

alternately,
As BA is to AD
EA is to AG [V.16]

In ▱ABCD, EG the sides about the common ∡BAD are proportional.

GF ∥ DC
∡AFG = ∡DCA

∡DAC is common to △ADC, △AGF
△ADC equiangular △AGF

For the same reason,
△ACB equiangular △AFE
▱ABCD equiangular ▱EG

Proportionally,
As AD is to DC
AG is to GF

As DC is to CA
GF is to FA

As CA is to CB
FA is to EA

As CB is to BA
FE is to EA

Proved that,
As DC is to CA
GF is to FA

As AC is to CB
AF is to FE

ex aequali,
As DC is to CB
GF is to FE [V.22]

In ▱ABCD, EG the sides about the equal angles are proportional,
▱ABCD is similar to ▱EG [VI.def.1]

For the same reason,
▱ABCD is similar to ▱HK

▱EG, ▱HK are similar to ▱ABCD
figures similar to the same figure are similar to one another, [VI.22]
▱EG is similar to ▱HK

QED

Proposition 25

to construct one and the same figure similar to a given rectilineal figure and equal to another rectilineal figure.

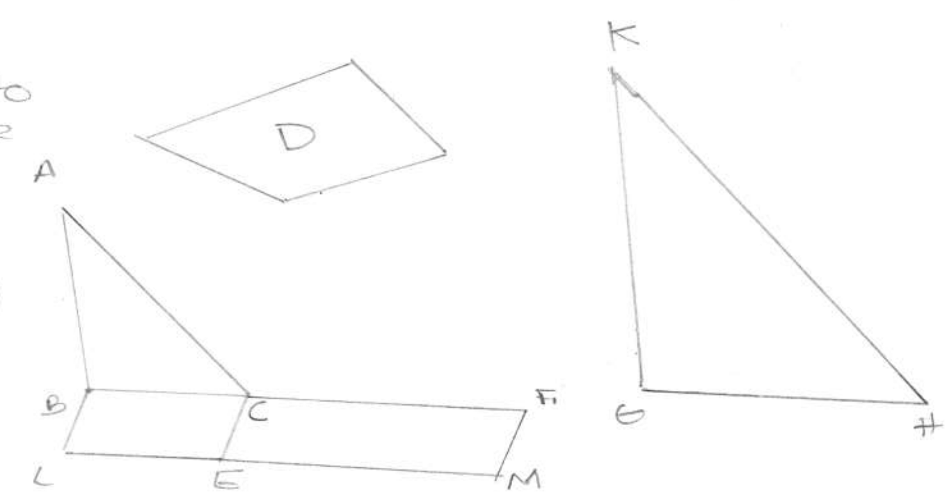
given.

rectilineal figure ABC to which the figure to be constructed must be similar

rectilineal figure D, to which the figure should be equal.

required.

one and the same figure similar to ABC, and equal to D.



Applied to $\overline{BC}$,

$\square BE = \triangle ABC$ [I.44]

Apply to CE

$\square CM = D$

in $\angle FCE = \angle CBL$ [I.45]

BC in straight line with CF

LE in straight line with EM

GH mean proportional to BC, CF [VI.13]

on GH

KGH similar and similarly situated to ABC [VI.18]

AS BC IS TO GH

GH IS TO CF

If three straight lines be proportional, as the first is to the third, so is the figure on the first to the similar and similarly situated figure on the second. [VI.19. Por]

AS BC IS TO CF

$\triangle ABC$ IS TO $\triangle KGH$.

AS BC IS TO CF

$\square BE$ IS TO $\square EF$ [VI.1]

as $\triangle ABC$ is to $\triangle KGH$

$\square BE$ IS TO $\square EF$

alternately,

$\triangle ABC$ IS TO $\square BE$

$\triangle KGH$ IS TO $\square EF$ [V.16]

$\triangle ABC = \square BE$

$\triangle KGH = \square EF$

$\square EF = D$

$\triangle KGH = D$

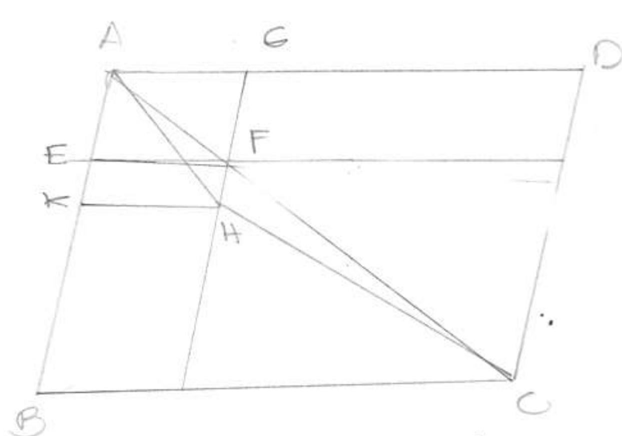
$\triangle KGH$ is similar to to ABC.

KGH has been constructed similar to ABC and equal to D

Q.E.D.

Proposition 26

If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it; it is about the same diameter with the whole.



given:

from ▱ABCD there
be taken away ▱AF
similar and similarly
situated.
∡DAB common

required:

▱ABCD is about the
same diameter with ▱AF

→suppose
AHC is the diameter of ▱ABCD
$\overline{GF} \rightarrow H$
$\overline{HK}$
HK ∥ AD, BC [I.31]

ABCD is about the same
diameter as KG
AS DA is to AB
GA is to AE

AS GA is TO AK
GA is TO AE [V.11]

GA has the same ratio to
AK, AE

AK = AE [V.9]
IMPOSSIBLE.

ABCD cannot be but about the same diameter to AF;
ABCD has the same diameter as AF.

Q.E.D.

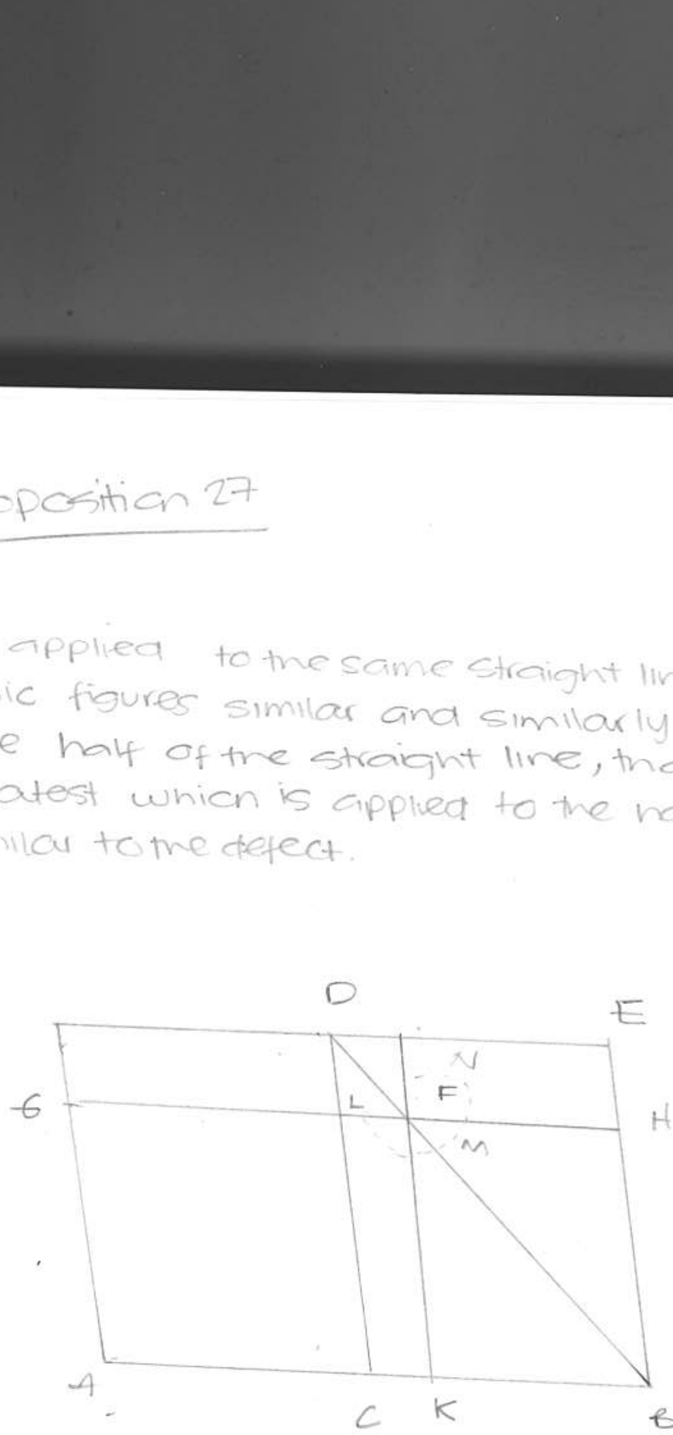
Proposition 27

Of all the parallelograms applied to the same straight line and deficient by parallelogramic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is the greatest which is applied to the half of the straight line and is similar to the defect.

given:

$\overline{AB}$ bisected at C.

Applied to AB ▱AD, deficient by parallelogrammic figure DB, described on half of AB, that is, CB.



required:

Of the parallelograms applied to AB and deficient parallelogramic figures similar and similarly situated to DB, AD is greatest.

Apply to AB
▱AF deficient by parallelogrammic figure FB, similar and similarly situated to DB

→ AD > AF

▱DB is similar to ▱FB
They are about the same diameter [VI.26]

diameter DB

CF = FE [I.43]
FB common

CH = KE

CH = CG
AC = CB [I.36]

GC = KE

→ add CF

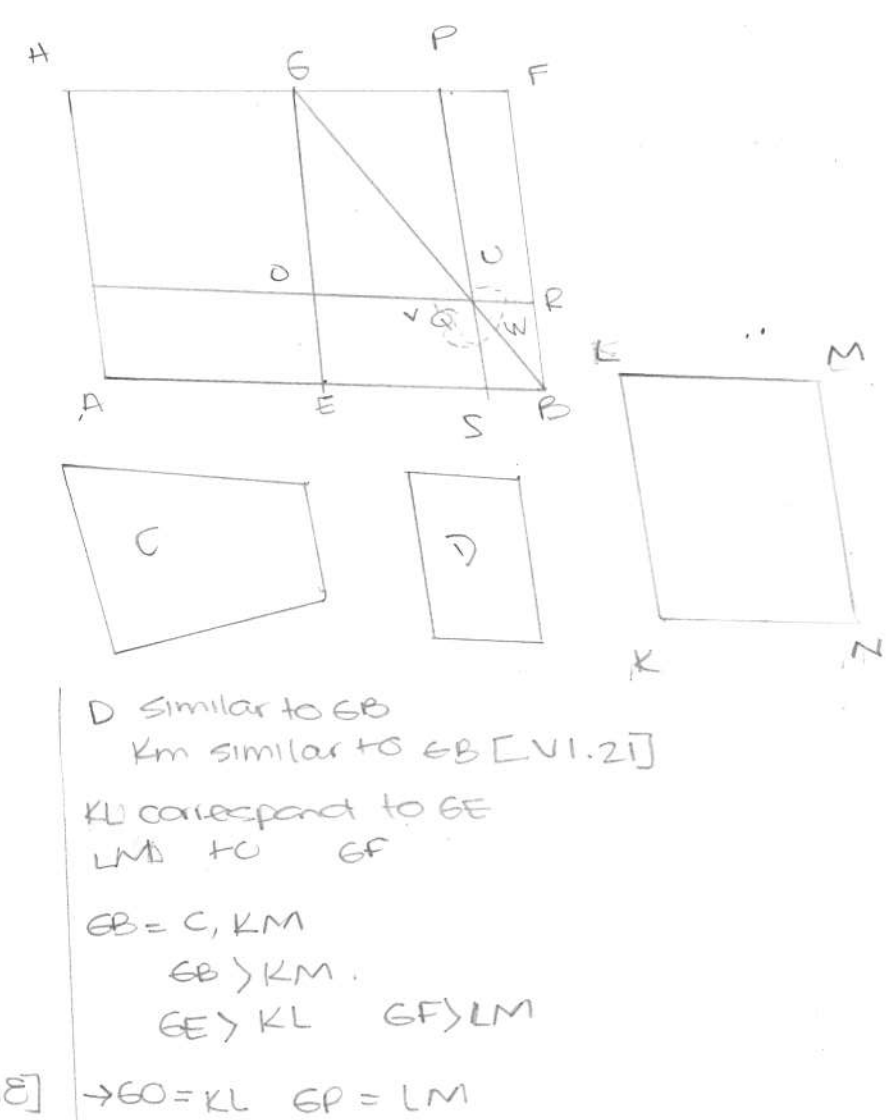
AF = gnomon LMN

▱AB (AD) > AF

Q.E.D.

Proposition 28

To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one: Thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.



given

AB. rectilineal figure C to which the figure to be applied to AB is required to be equal. Not greater than the parallelogram on the half of AB and similar to the defect.
D the figure to which the defect is required to be similar.

required

Apply to AB a parallelogram equal to C and deficient by parallelogramic figure similar to D

AB bisected at E
on EB
EBFG similar and similarly situated to D [VI.18]

▱AG complete

If AG = C
There has been applied to AB ▱AG = C and deficient by figure GB similar to D

→If not,
HE > C

HE = GB
GB > C

Construct KLMN = to the excess by wich GB is greater than C and similar and similarly situated to D [VI.25]

D similar to GB
KM similar to GB [VI.21]

KL correspond to GE
LM to GF

GB = C, KM
GB > KM.
GE > KL   GF > LM

→GO = KL   GP = LM

▱OGPQ
▱OGPQ =/similar to KM

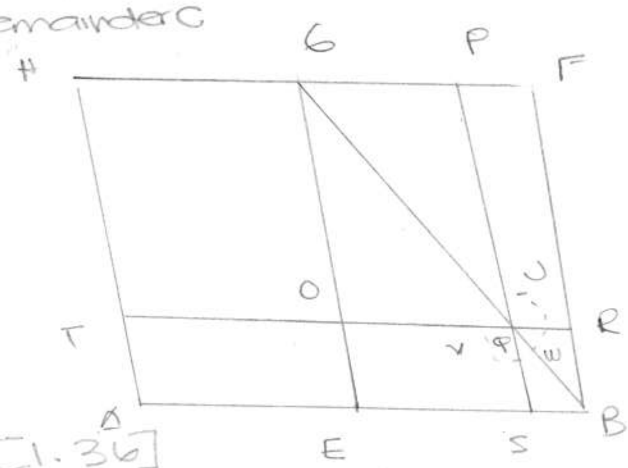
QG similar to GB   [VI.21]
GQ is in same diameter to GB [VI.26]

GQB be the diameter

→

BE = C,KM
GQ = KM

gnomon UWV = remainder C

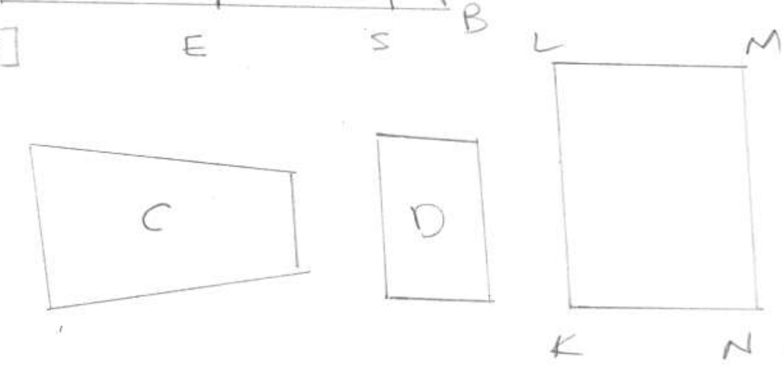


PR = OS
→add QB

PB = OB

OB = TE
side AE = side EB [1.36]
TE = PB

→add OS
TS = gnomon UWV



gnomon UWV = C
TS = C

given straight line AB there has been applied the parallelogram ST equal to the given rectilineal figure C and deficient by a parallelogramic figure QB which is similar to D

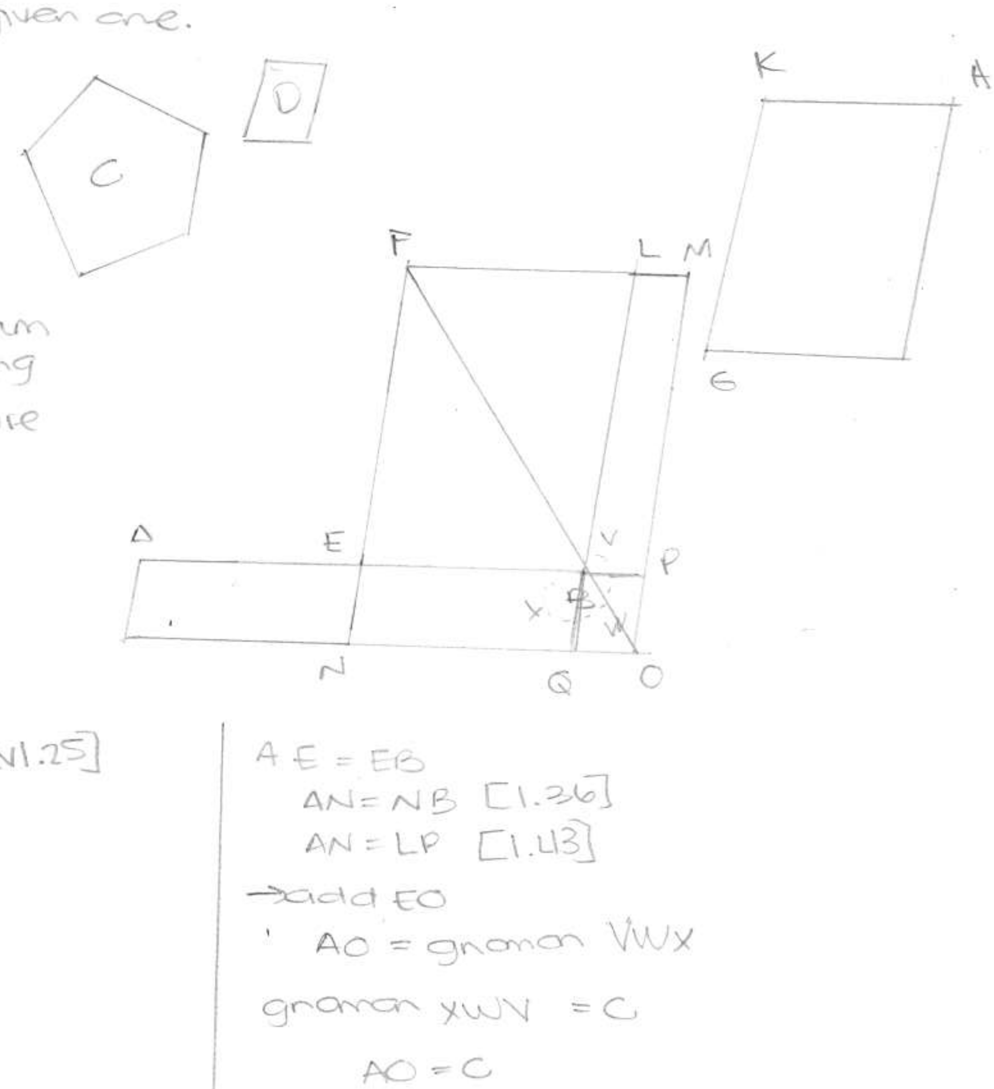
Q.E.F

Proposition 29

to a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one.

given.
$\overline{AB}$
rectilineal figure C
D

required.
Apply to $\overline{AB}$ a parallelogram equal to C and exceeding by a parallelogrammic figure similar to D



$\overline{AB}$ bisected at E
on EB
▱ similar and similarly situated to D

GH = BF + C similar and similarly situated to D [VI.25]

KH corresponds to FL
KG to FE

GH > FB
KH > FL , KG > FE

$\overline{FE}, \overline{FL}$ →
FLM = KH
FEN = KG
▱MN completed
MN equal / similar GH

GH similar to EL
MN similar to EL [VI.21]
EL about the same diameter as MN [VI.26]
diameter FO be drawn

GH = EL, C
GH = MN
MN = EL, C
→ subtract EL
gnomon XWV = C

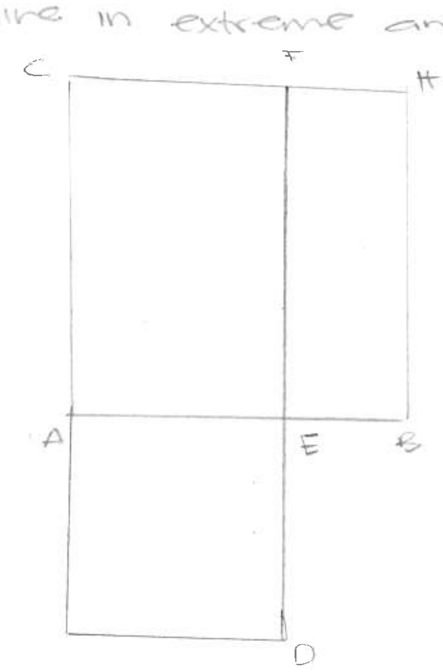
AE = EB
AN = NB [I.36]
AN = LP [I.43]
→ add EO
∴ AO = gnomon VWX
gnomon XWV = C
AO = C

given $\overline{AB}$ there has been applied a parallelogram AO equal to the given rectilineal figure C and exceeding by a parallelogrammic figure QP which is similar to D, since PQ is also similar to EL [VI.24]

Q.E.F.

Proposition 30

to cut a given finite straight line in extreme and mean ratio.



given:

AB

required:

cut AB in extremes and mean ratio

on AB
square BC

applied to AC
▱CD = BC
exceeding by the figure
AD similar to BC
[VI.29]

BC is a square
AD is a square

BC = CD
→subtract CE
BF = AD

It is also equiangular
in BF, AD the sides about the equal angle are reciprocally proportional [VI.14]

AS FE IS TO ED
AE IS TO EB

FE = AB
AD = AE

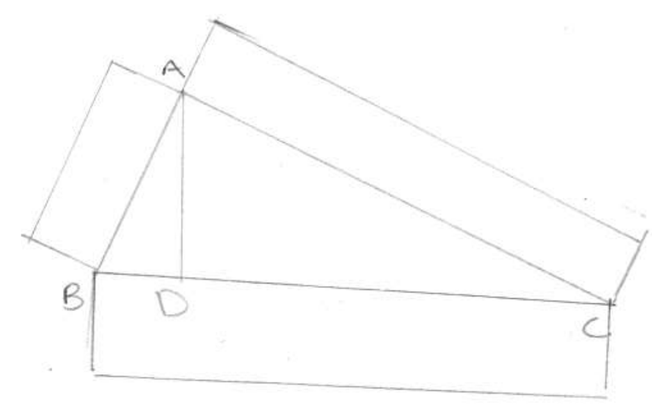
AS BA IS TO AE
AE is to EB

AB > AE
AE > EB

Straight line AB has been cut in extremes and mean ratio at E, and the greater segment of it is AE.

Proposition 31

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.



given:

$\triangle ABC$ right-angled

$\angle BAC = \llcorner$

required:

Figure on BC is equal to the similar and similarly described figures on BA, AC

$\overline{AD}$ perpendicular

In right-angled $\triangle ABC$
AD has been drawn from the right $\angle$ $\triangle$ perpendicular to base BC

$\triangle ABD$, $\triangle ADC$ adjoining the perpendicular are similar to $\triangle ABC$ and to each other [VI.8]

$\triangle ABC$ is similar to $\triangle ABD$

As CB is to BA
AB is to BD [VI.def.1]

three straight lines are proportional,
as the first to the third, so is the first similar and similarly described figure on the second. [VI.19.Por]

As CB is to BD

CB is similar and similarly described to BA.

For the same reason,
As BC is to CD
figure BC is to figure CA

As BC is to BD
Figure BC to the similar and similarly described figure on BA, AC
[V.24]

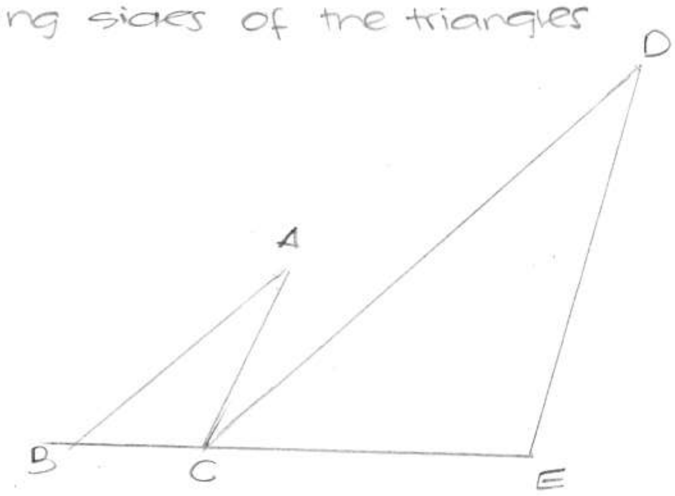
BC = DB, DC;

figure on BC is equal and similarly described figures on BA, AC.

Q.E.D

Proposition 32

If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line.



given:

$\triangle ABC$, $\triangle DCE$
sides BA, AC proportional to DC, DE
As AB is to AC
DC is to DE

$AB \parallel DC$, $AC \parallel CE$

required:

BC is in straight line with CE

$AB \parallel CD$
$\overline{AC}$ has fallen upon them
$\angle BAC = \angle ACD$ [I.29]

For the same reason,
$\angle CDE = \angle ACD$

$\angle BAC = \angle CDE$

$\triangle ABC$, $\triangle CDE$ are two $\triangle$ having one angle A equal to another angle D.
The sides about the equal $\angle$ are proportional

As BA is to AC
CD is to DE
$\triangle ABC$ equiangular $\triangle CDE$ [VI.6]
$\angle ABC = \angle DCE$

$\angle ACD = \angle BAC$
$\angle ACE = \angle ABC, \angle BAC$
$\rightarrow$ add $\angle ACB$
$\angle ACE, \angle ACB = \angle ABC, \angle BAC, \angle ACB$

$\angle BAC, \angle ABC, \angle ACB = 2\llcorner$ [I.32]
$\angle ACE, \angle ACB = 2\llcorner$

with $\overline{AC}$, and at C on it, 2 straight lines BC, CE not lying on the same side make the adjacent angles $\angle ACB$, $\angle ACE$ equal to two right angles

BC is in straight line with CE [I.14]

QED

Proposition 33

In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.

given

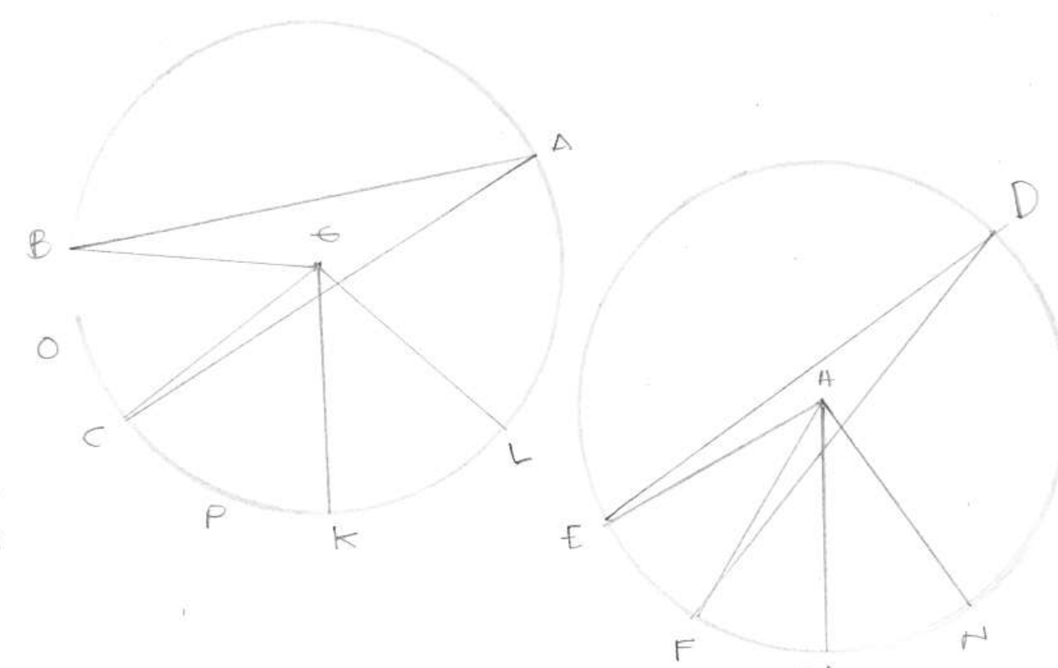
$\bigcirc ABC = \bigcirc DEF$

$\measuredangle BGC$, $\measuredangle EHF$ be at their centres G, H.

$\measuredangle BAC$, $\measuredangle DEF$ at their circumference

required

As circumference BC is to circumference EF, so is $\measuredangle BGC$ to $\measuredangle EHF$, and $\measuredangle BAC$ to $\measuredangle EDF$



let any number of consecutive circumferences CK, KL be made equal to circumference BC

any number of consecutive circumferences FM, MN equal to circumference EF

$\overline{GK}, \overline{GL}.. \overline{HM}, \overline{HN}$

BC = CK = CL

$\measuredangle BGC = \measuredangle CGK = \measuredangle KGL$ [III.27]

whatever multiple BL is o BC, that multiple is also $\measuredangle BGL$ of $\measuredangle BGC$

For the same reason, whatever multiple NE is of EF, that multiple is $\measuredangle NHE$ of $\measuredangle EHF$.

If BL = EN

$\measuredangle BGL = \measuredangle EHN$ [III.27]

If BL > EN

$\measuredangle BGL > EHN$

if less, less.

There being four magnitudes, two circumferences BC, EF and two angles BGC, EHF

There have been taken of BC and $\measuredangle BGC$ equimultiples BL, $\measuredangle BGL$

and of EF and $\measuredangle EHF$ equimultiples EN, $\measuredangle EHN$

It was also proved that

If BL is in excess of EN

$\measuredangle BGL$ is in excess of $\measuredangle EHN$

if equal, equal

if less, less

As BC is to EF

$\measuredangle BGC$ is to $\measuredangle EHF$ [V.def.5]

As $\measuredangle BGC$ is to $\measuredangle EHF$

$\measuredangle BAC$ is to $\measuredangle EDF$

for they are doubles respectively.

As BC is to EF

$\measuredangle BGC$ is to $\measuredangle EHF$

$\measuredangle BAC$ is to $\measuredangle EDF$

QED.