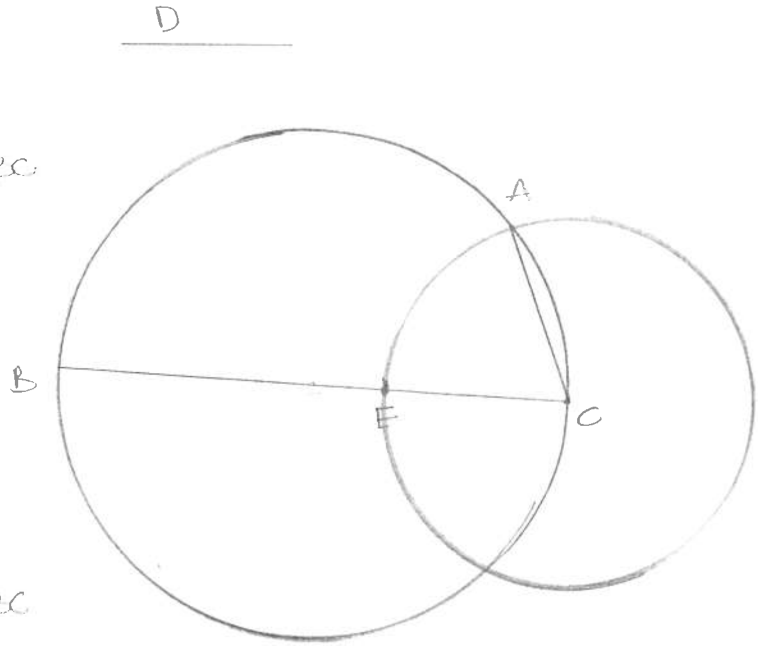
Proposition 1

Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.



given

○ABC

$\overline{D}$ < diameter of ○ABC

required

fit into ○ABC a straight line $= \overline{D}$

diameter $\overline{BC}$ of ○ABC

→ IF BC = D

BC has been fitted into ○ABC

= D

→ IF BC > D

CE = D

○EAF, centre C, distance CE

$\overline{CA}$

C is centre of ○AEF

CA = CE

CE = D

CA = D

given ○ABC there was fitted $\overline{CA} = \overline{D}$

QEF

Proposition 2

In a given circle to inscribe a triangle equiangular with a given triangle.

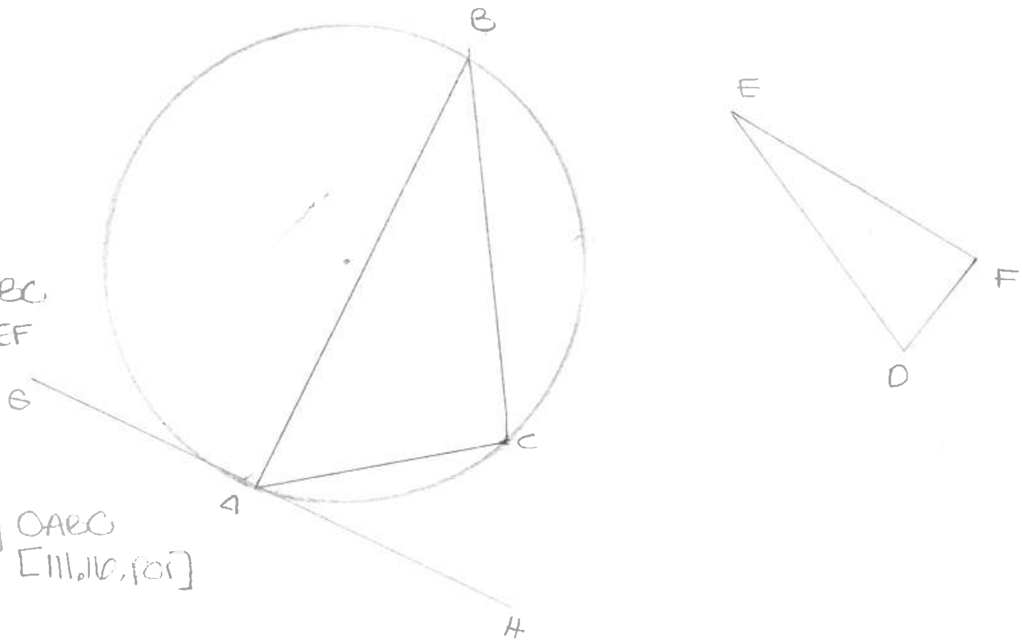
given:

⊙ABC

△DEF

required:

inscribe in ⊙ABC
a △ with △DEF



$\overline{GH}$ touching ⊙ABC
at A [III.16, Por]

∡HAC = ∡DEF
on A [I.23]

∡GAB = ∡DFE
on A

$\overline{BC}$

AH touches ⊙ABC
From A, $\overline{AC}$ is drawn across

∡HAC = ∡ABC
in alternate segment [III.32]

∡HAC = ∡DEF

∡ABC = ∡DEF

For the same reason,

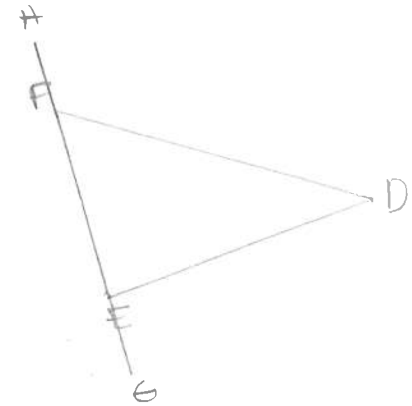
∡ACB = ∡DFE

∡BAC = ∡EDF [I.32]
(remaining ∡)

In a given circle there has been inscribed a △ = △ Q.E.F

Proposition 3

About a given circle to circumscribe a triangle equiangular with a given triangle.



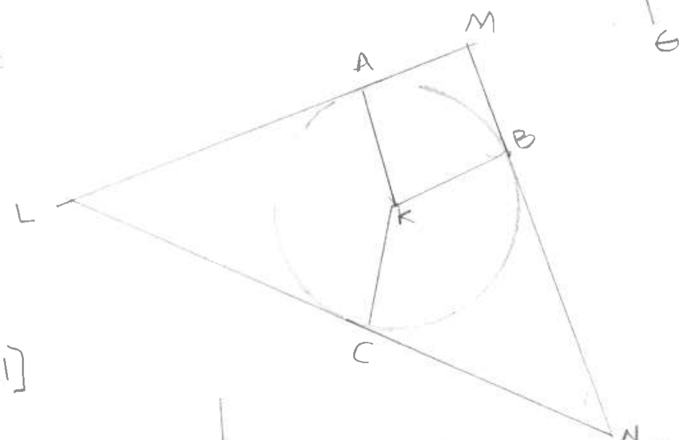
given

⊙ABC

ΔDEF

Required

Circumscribe about ⊙ABC a Δ equiangular to ΔDEF



EF → G

EF → H

Centre K of ⊙ABC [III.1]

KB

∡BKA = ∡DEG
∡BKC = ∡DFH [I.23]

LAM through A
MBN through B [III.16 por]
NCL through C
Touching ⊙ABC

LM, MN, NL touch ⊙ABC
KA, KB, KC joined from centre
∡ at A, B, C are ⊾ [III.18]

All ∡ of ▱AMBK = ⊾
▱AMBK is divisible in 2Δ

∡KAM, ∡KBM = 2⊾
∡AKB, ∡AMB = 2⊾

∡DEG, ∡DEF = 2⊾ [I.13]

∡KAM, ∡KBM = ∡DEG, ∡DEF

∡AKB = ∡DEG

∡AMB = ∡DEF

Similarly,

∡LNB = ∡DFE

∡MLN = ∡EDF [I.32]

ΔLMN is equiangular to ΔDEF and is circumscribed about ⊙ABC

Q.E.F.

Proposition 4

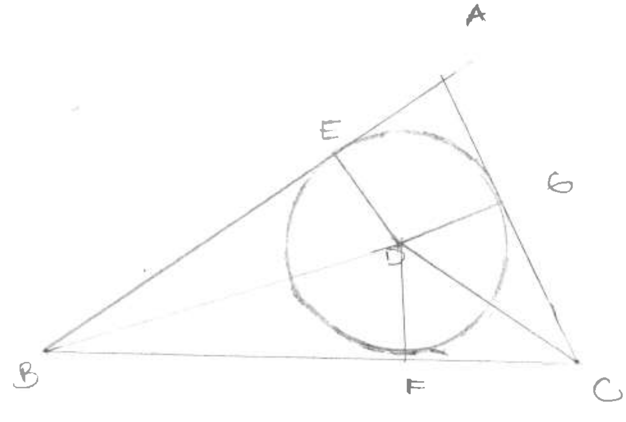
In a given triangle to inscribe a circle

given

$\triangle ABC$

required:

inscribe a ○ in $\triangle ABC$



bisect ∡ABC by $\overline{BD}$
bisect ∡ACB by $\overline{CD}$ [I.9]

$\overline{DE}, \overline{DF}, \overline{DG}$ perpendicular to $\overline{AB}, \overline{BC}, \overline{CA}$.

∡ABD = ∡CBD

∡BED = ⊾
∡BED = ∡BFD

$\triangle EBD = \triangle FBD$
2∡ = 2∡
side = side
BD common
remaining sides = remainig sides
[I.26]

DE = DF

For the same reason,
DG = DF

DE = DF = DG

○ centre D, distance DE will pass through DF, DG and touch $\overline{AB}, \overline{BC}, \overline{CA}$
For the ∡ is right

For, if it cuts them,
The straight line drawn at ⊾ to the diameter of ○ from its extremity will be within the circle
ABSURD [III.16

circle with centre D and distance of one of the lines DE, DF, DG will not cut the lines AB, BC, CA,
Therefore it will touch them, and will be the circle inscribed in $\triangle ABC$
[IV DEF.5]

let it be inscribed as FGE

Q.E.F

Proposition 5

-About a given triangle to circumscribe a circle.

Given:

ΔABC

equired:

ircumscribe a ○
bout ΔABC

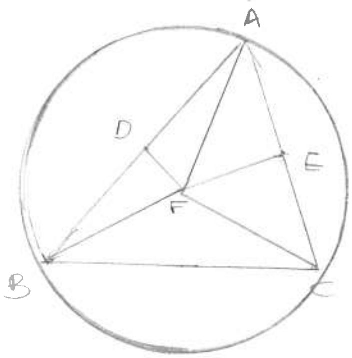


figure A

isect $\overline{AB}$, $\overline{AC}$ at D, E [1.10]

DF, EF at ⊥ $\overline{AB}$, $\overline{AC}$

They will meet within ΔABC,
on BC, outside $\overline{BC}$

WITHIN [Figure A]

$\overline{FB}$, $\overline{FC}$, $\overline{FA}$

AD = DB
DF is common at ⊥
Base AF = Base FB [1.4]

Similarly,
CF = AF
FB = CF
FA = FB = CF

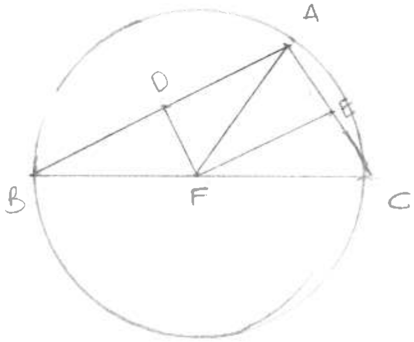
○ centre F
distance FA, FB or CF
will pass through the
remaining points

Circle will be circumscribed
about ΔABC

ON BC

DE, EF meet on BC at F

$\overline{AF}$



Similarly,

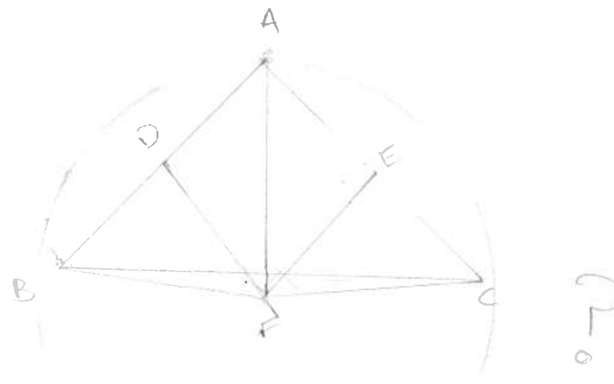
F centre of ⊙ABC

Circumscribed about △ABC

outside BC

DF, EF meet outside △ABC at F

$\overline{AF}$, $\overline{BF}$, $\overline{CF}$



AD = DB

DF common at ⊾

Base AF = Base BF [I 4]

Similarly,

CF = AF

BF = FC

described with centre F and distance FA, FB, FC will pass also through the remaining points, circumscribed about △ABC

Q.E.F

When the centre of the circle falls within the triangle, the ∡BAC, being in a segment greater than the semicircle, is less than a right ∡;

when the centre falls on the straight line BC, ∡BAC, being in a semicircle is right;

and when the centre of the circle falls outside the triangle, ∡BAC, being in a segment less than a semicircle, is greater than a right ∡.

[III.31]

Proposition 6

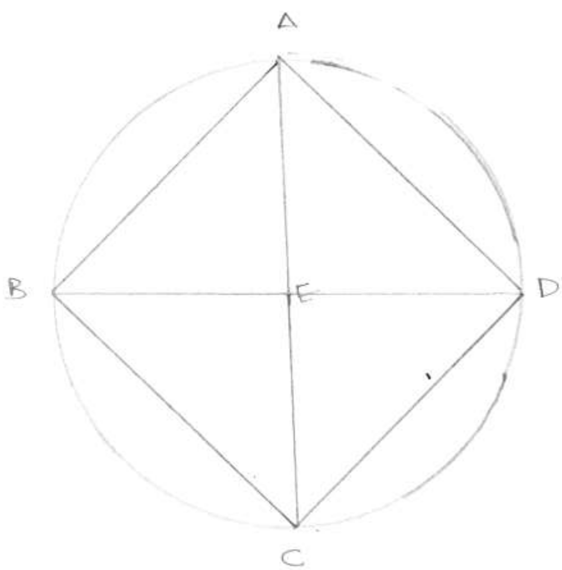
In a given circle to inscribe a square.

given.

⊙ABCD

equired

nscribe a square
n ⊙ABCD



diameters $\overline{AC}, \overline{BD}$ of ⊙ABCD
drawn at ∟

$\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$

E = centre
E = ED
EA is common
base BA = base AD [I.4]

or the same reason,
BC = AB
CD = AD
BCD is equilateral

it is also right-angled
BD is diameter of ⊙ABCD
BAD is semicircle
∡BAD = ∟ [III.31]

ame reason,
∡ABC = ∟
∡BCD = ∟
∡CDA = ∟

quadrilateral ABCD is ∟ angled

But it is also equilateral

ABCD is a square [I. Def. 22]
and is inscribed in ⊙ABCD

Q.E.F.

Proposition 7

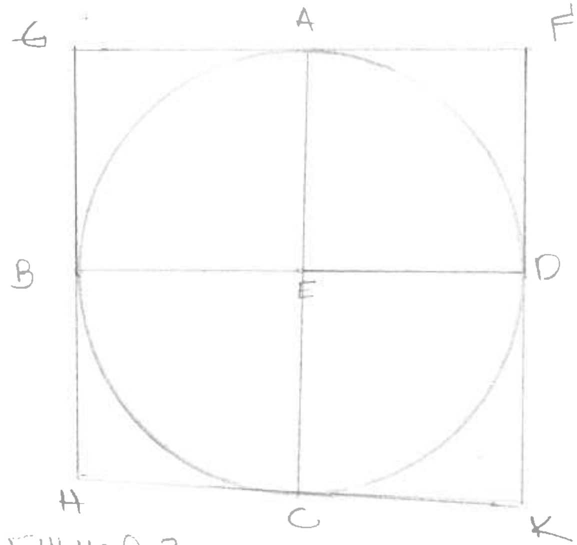
about a given circle to circumscribe a square.

given:

⊙ABCD

required:

circumscribe a square about ⊙ABCD



diameters AC, BD drawn at ∟ to each other

at A, B, C, D let $\overline{FG}, \overline{GH}, \overline{HK}, \overline{KF}$ be drawn touching ⊙ABCD [III.16, Por]

FG touches ⊙ABCD
EA joined from centre to A
∡A = ∟ [III.18]

For the same reason,
∡B = ∟
∡C = ∟
∡D = ∟

∡AEB = ∟
∡EBG = ∟
GH ∥ AC [I.28]

For the same reason,
AC ∥ FK
GH ∥ FK [I.30]

Similarly,
GF, HK ∥ BED

GK, GC, AK, FB, BK are parallelograms
GF = HK
GH = FK [I.34]

AC = BD
AC = GF, HK
BD = GF, HK [I.34]
FGHK is equilateral

→ right-angled
GBEA = ▱
∡AEB = ∟
∡AGB = ∟ [I.34]

Similarly,
∡H = ∟, ∡K = ∟, ∡F = ∟

FGHK = ∟ angled

But it was also proven equilateral

FGHK = a square
circumscribed around ⊙ABCD

Q.E.F

Proposition 8

In a given square to inscribe a circle.

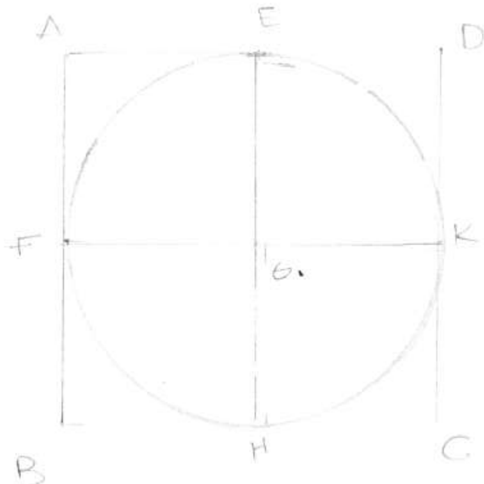
given:

Square ABCD

required

inscribe a O

in square ABCD



bisect AD at E [1.10]

bisect AB at F

EH ll AB, CD [1.31]

FK ll AD, BC

AK, KB, AH, HD, AG, GC, BG, GD - //

Their opposite sides are equal [1.34]

AD = AB

AE = Half AD

AF = Half AB

AE = AF

opposites are equal

FG = GE

similarly,

GH, GK = FG, GE

GH = GK = GE = FG

O with centre G

distance GH, GK, GE or GF

will touch AB, BC, CD, DA

at right ∢

For, IF

The circle cuts AB, BC, AD, DC

the straight line drawn at right ∢ to the diameter of the circle from its extremity will fall within the circle

↳ ABSURD [III.16]

Circle with centre G and distance GH, GK, GE or GF will not cut the straight lines AB, BC, CD, DA

It will touch them and be inscribed in square ABCD

Q.E.F

Proposition 9

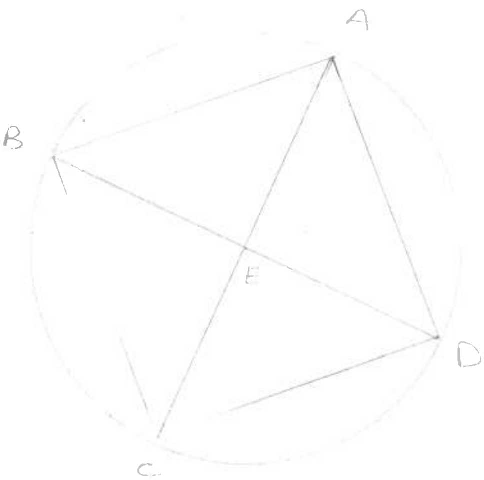
About a given square to circumscribe a circle

given

square ABCD

required

circumscribe a circle about square ABCD



$\overline{AC}, \overline{BD}$ cutting each other at E

DA = AB

AC is common

DA, AC = AB, AC

base DC = base BC

∡DAC = ∡BAC [1.8]

∡DAB is bisected by AC

Similarly,

∡ABC, ∡BCD, ∡CDA are bisected by AC, DB

∡DAB = ∡ABC

∡EAB is half ∡DAB

∡EBA is half ∡ABC

∡EAB = ∡EBA

EB = EA [1.6]

similarly,

EA, EB = EC, ED

EA = EB = EC = ED

○ centre E

distance EA, EB, EC or ED

will pass through the remaining points

and will be circumscribed about square ABCD

QEF

Proposition 10

To construct an isosceles triangle having each of the angles at the base double the remaining one.

given:

$\overline{AB}$

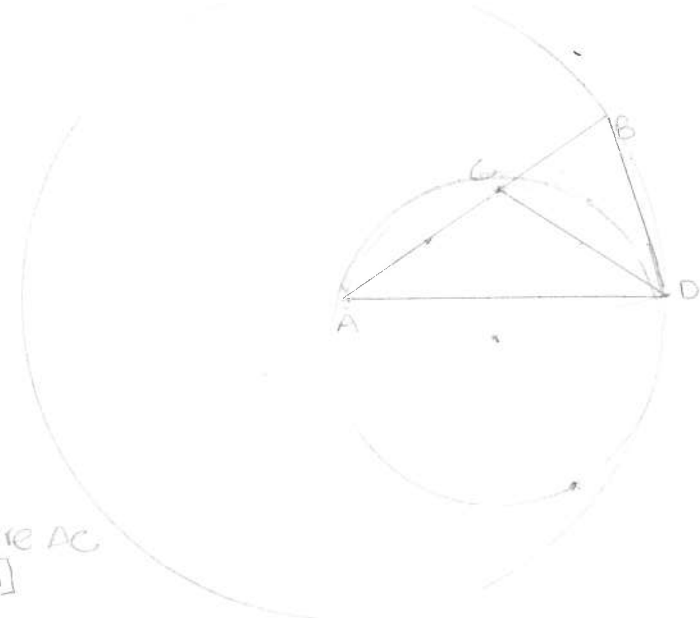
required

isosceles △

∡ at base

double the

remaining one



$\overline{AB}$ cut at C

rectangle AB, BC = square AC

∴ [II.11]

⊙BDE

centre A, distance AB

in ⊙BDE

$\overline{BD} = \overline{AC}$

AC is not > than diameter BDE [IV.1]

$\overline{AD}$, $\overline{DC}$

⊙ACD circumscribed about △ACD [IV.5]

rectangle AB, BC = square AC.

AC = BD

rectangle AB, BC = square BD

· B has been taken outside circle ACD

from B,

BA, BD have fallen on circle ACD

one cuts it and the other falls on it

rectangle AB, BC = square BD

BD touches ⊙ACD [III.37]

BD touches it

DC drawn across point of contact D

∡BDC = ∡DAC [III.32]

add ∡CDA

∡BDA = ∡CDA, ∡DAC

Exterior $\angle BCD = \angle CDA, \angle DAC$ [1.32]

$\angle BDA = \angle BCD$

$\angle BDA = \angle CBD$

$AD = AB$ [1.5]

$\angle DBA = \angle BCD$

$\angle BDA = \angle DBA, = \angle BCD$

$\angle DBC = \angle BCD$

$BD = DC$ [1.6]

$DB = AC$

$AC = CD$

$\angle CDA = \angle DAC$ [1.5]

$\angle CDA, \angle DAC = 2 \angle DAC$

$\angle BDC = \angle CDA, \angle DAC$

$\angle BDC = 2 \angle DAC$

$\angle BDC = \angle BDA, \angle DBA$

$\angle BDA = 2 \angle DAC$

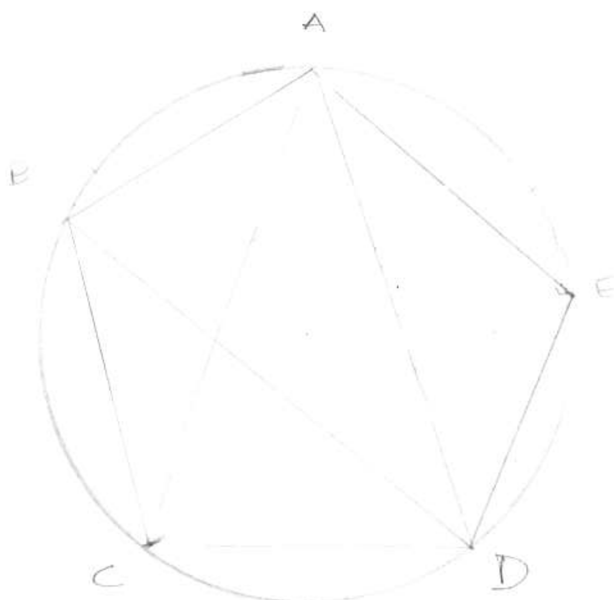
$\angle DBA = 2 \angle DAC$

An isoceles triangle has been constructed having each of the angles at the base DB double the remaining one.

QEF

Proposition 11

In a given circle to inscribe an equilateral and equiangular pentagon.



given

⊙ABCD

required

inscribe in ⊙ABCD an equiangular pentagon

isosceles △FGH

∡G, ∡F double ∡F [IV.10]

inscribe in ⊙ABCD

△ACD equiangular with △FGH

∡CAD = ∡F
∡ACD = ∡G
∡CDA = ∡H
[IV.2]

∡ACD, ∡CDA = 2∡CAD

bisect ∡ACD by CE
bisect ∡CDA by DB [I.9]

$\overline{AB}, \overline{BC}, \overline{DE}, \overline{EA}$

∡ACD, ∡CDA = 2∡CAD
and have been bisected by CE, DB

∡DAC = ∡ACE = ∡ECD = ∡CDB = ∡BDA
[III.26]

Angles stand on equal circumferences

AB = BC = CD = DE = EA [III.29]

Pentagon ABCDE is equilateral

→ equiangular

circumference AB = circumference DE

- add BCD

circumference ABCD = circumference EDCB

∡AED stands on circumference ABCD

∡BAE stands on circumference EDCB

∡BAE = ∡AED [III.27]

For the same reason,

∡ABC, ∡BCD, ∡CDE = ∡BAE, ∡AED

Pentagon ABCDE is equiangular

Pentagon ABCDE is equilateral and equiangular

Q.E.F.

Proposition 12

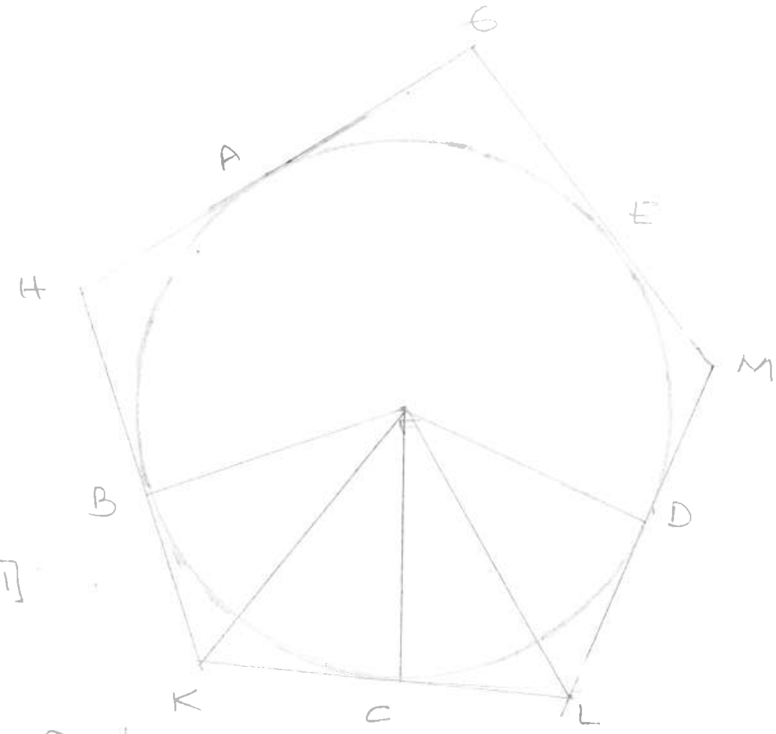
About a given circle to circumscribe an equilateral and equiangular pentagon

given

○ABCDE

required:

circumscribe an equilateral and equiangular pentagon in ○ABCDE



A, B, C, D, E angular points

circumferences AB, BC, CD, DE, EA are equal [IV.11]

Through A, B, C, D, E
GH, HK, KL, LM, MG touching ○ [III, 16, Por]

centre F [III.1]

FB, FK, FC, FL, FD

KL touches ○ABCDE at C
FC joined from centre to C
FC is perpendicular to KL
∡C = ∟ [III 18]

For the same reason,
∡B = ∟
∡D = ∟

∡FCK = ∟
square FK = squares FC, CK [I.47]

For the same reason,
square FK = squares KB, BF

squares FC, CK = squares KB, BF
square FC = square FB
square CK = square KB

BK = CK

FB = FC
FK common

BF, FK = CF, KF
base BK = Base CK
∡BFK = ∡KFC [I.8]
∡BKF = ∡CKF

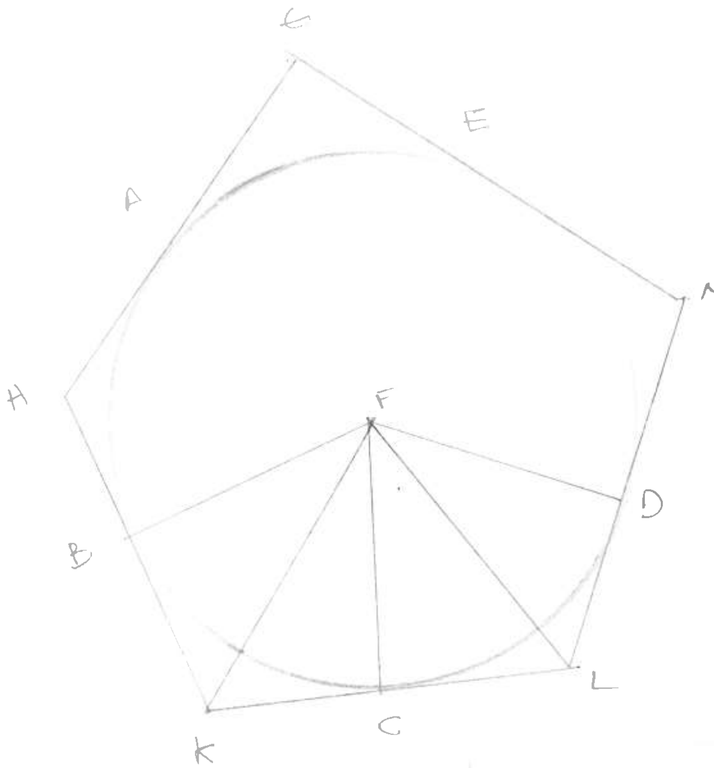
∡BFC = 2∡KFC
∡BKC = 2∡CKF

For the same reason,
∡CFD = 2∡CFL
∡DLC = 2∡FLC

circumference BC = CD
∡BFC = ∡CFD [III.27]

∡BFC = 2∡KFC
∡CFD = 2∡LFC
∡KFC = ∡LFC

∡FCK = ∡FCL



ΔFKC, ΔKLC have 2∡ = 2∡
side = side ; FC common
The remaining sides = remaining sides
remaining ∡ = remaining ∡
[I.26]

KC = CL
∡FKC = ∡FLC

KC = CL
KL = 2KC

For the same reason,
HK = 2BK

BK = KC
HK = KL

similarly,
HG, GM, ML = HK, KL
pentagon GHKLM is equilateral

→ equiangular
∡FKC = ∡FLC
∡HKL = 2∡FKC
∡KLM = 2∡FLC
∡HKL = ∡KLM

Similarly,
∡KHG = ∡HGM = ∠GML = ∡HKL = ∡KLM

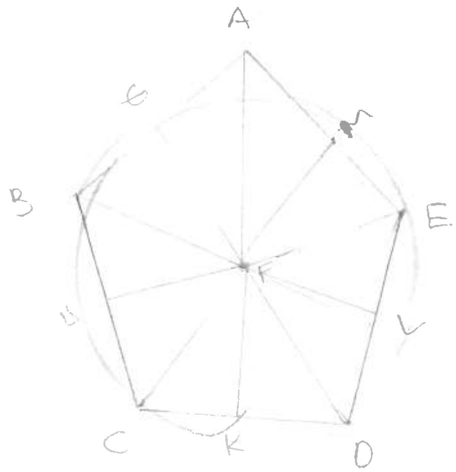
Pentagon GHKLM is equiangular.

Pentagon GHKLM is
equilateral and
equiangular

QEF

Proposition 13

In a given pentagon, which is equilateral and equiangular, to inscribe a circle.



?

given:

Pentagon ABCDE
equiangular and
equilateral

required:

inscribe a circle in
Pentagon ABCDE

bisect ∡BCD by CF
bisect ∡CDE by DF
CF, DF meet at F
FB, FA, FE

BC = CD
CF common
BC, CF = DC, CF
∡BCF = ∡DCF
base BF = base DF [I.4]
remaining sides = remaining sides
remaining ∡ = remaining ∡
∡CBF = ∡CDF

∡CDE = 2∡CDF
∡CDE = ∡ABC

∡CBA = 2∡CBF
∡ABF = ∡FBC
∡ABC is bisected by BF

Similarly,
∡BAE bisected by FA
∡AED bisected by FE.

FG, FH, FK, FL, FM perpendicular to
AB, BC, CD, DE, EA

∡HCF = ∡KCF
∡FHC = ∟
∡FHC = ∡FKC

△FHC, △KFC 2 angles = 2 angles
one side = one side (FC common)
remaining sides = remaining sides
[I.26]

FH = FK

Similarly,
FL = FH = FK = FL, FM

○ centre F
distance FG, FH, FK, FL, FM
will pass through the remaining points.
will touch AB, BC, CD, DE, EA
∡G, ∡H, ∡K, ∡L, ∡M = ∟

For, if it does not touch them, but cuts them, the straight line drawn at ∟ to the diameter of the ○ from its extremity falls within the circle
↳ ABSURD [III.16]

Circle centre F, distance FG, FH, FK, FL, FM
will touch AB, BC, CD, DE, EA

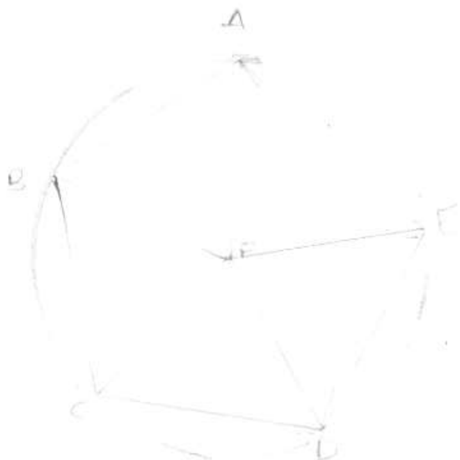
In a given pentagon a circle is inscribed Q.E.F.

Proposition 14

About a pentagon, which is equilateral and equiangular, to circumscribe a circle.

Given:

Pentagon ABCDE, equiangular and equilateral



Required:

Circumscribe a circle about Pentagon ABCDE

bisect ∡BCD by CF
bisect ∡CDE by DF
FB, FA, FE

∡BCD = ∡CDE
∡FCD = half ∡BCD
∡CDF = half ∡CDE

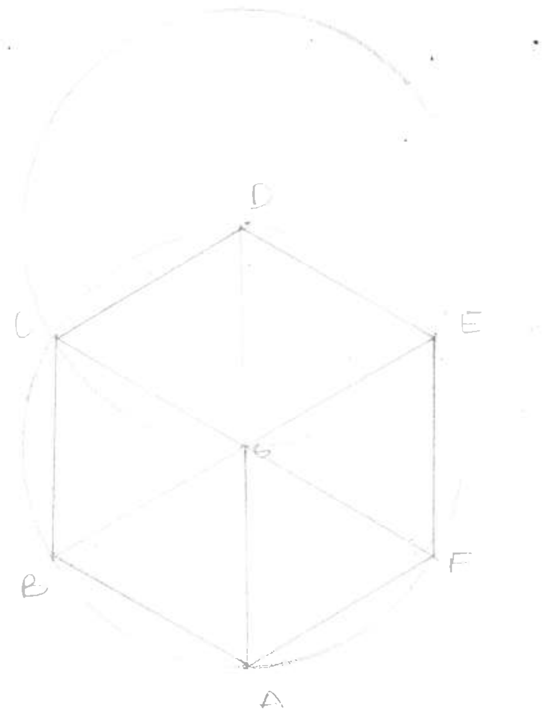
∡FCD = ∡CDF
FC = FD [1.6]

Similarly,
FA = FB = FC = FD = FE

O centre F
distance FA, FB, FC, FD or FE
will pass through the remaining points and be circumscribed in pentagon ABCDE

Proposition 15

In a given circle to inscribe an equilateral and equiangular hexagon.



given:

○ ABCDEF

required

equilateral and equiangular hexagon inscribed in ○ABCDEF

diameter AD

centre G

○EGCH, centre D, distance DG

$\overline{EG}$, $\overline{GC}$

$\overline{EG} \to B$

$\overline{GC} \to F$

$\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$

G is centre of ○ABCDEF

GE = GD

D is centre of ○EGCH

DE = DG

GE = DE

△EGD is equilateral

∡EGD = ∡GDE = ∡DEG [I.5]

3 ∡ of a △ = 2⊾ [I.32]

∡EGD = 1/3 2⊾

Similarly,

∡DGC = 1/3 2⊾

straight line GC standing on EB makes ∡EGC, ∡CGB = 2⊾

∡CGB = 1/3 2⊾

∡EGD = ∡DGC = ∡CGB

angles vertical to them are equal [I.15]

∡EGD = ∡DGC = ∡CGB = ∡BGA = ∡AGF = ∡FGE

But the ∡ stand on equal circumferences [III.26]

circumferences AB, BC, CD, DE, EF, FA equal one another.

equal circumferences are subtended by equal straight lines [III.29]

six straight lines equal one another

hexagon ABCDEF is equilateral

→ Equiangular

circumference FA = circumference ED

→ add circumference ABCD

circumference FABCD = circumference EDCBA

∡FED is on circumference FABCD

∡AFE is on circumference EDCBA

∡FED = ∡AFE [III.27]

Similarly,
remaining angles of hexagon ABCDEF = ∡AFE, ∡FED

hexagon ABCDEF is equiangular

It was also equilateral
and is inscribed in ○ABCDEF

Q.E.F

PORISM

From this it is manifest that the side of the hexagon is equal to the radius of the circle

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it

Q.E.F

Proposition 16.

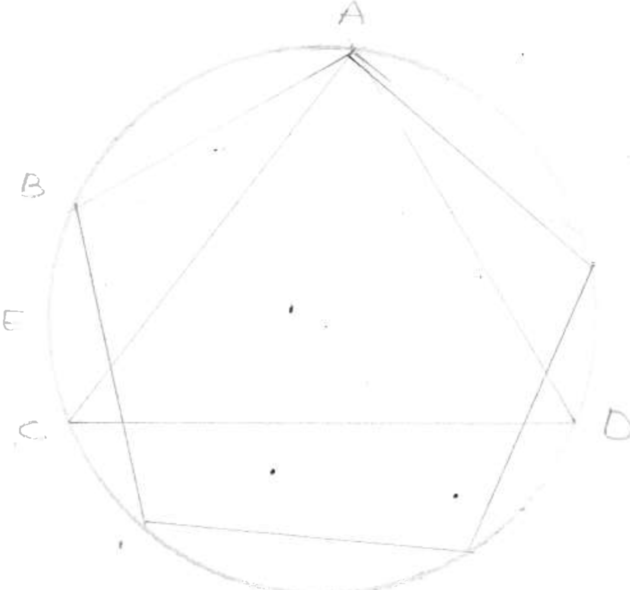
In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.

given:

OABCD

required

inscribe in OABCD a fifteen-angled figure, equilateral and equiangular



$\overline{AC}$ of an equilateral Δ

$\overline{AB}$ of equilateral pentagon

5 equal segments in circumference ABC (1/3 of O)

3 equal segments in circumference AB (1/5 of O)

bisect $\overline{BC}$ at E [III.30]

circumference BE, EC (1/15 of O)

If we join BE, EC and fit into OABCD straight lines equal to them and in contiguity, a fifteen-angled figure both equilateral and equiangular will have been inscribed in it

Q.E.F.

CORollARY:

And, in like manner as is the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equilateral.

And, further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen-angled and circumscribe one about it. Q.E.F.