Proposition 1

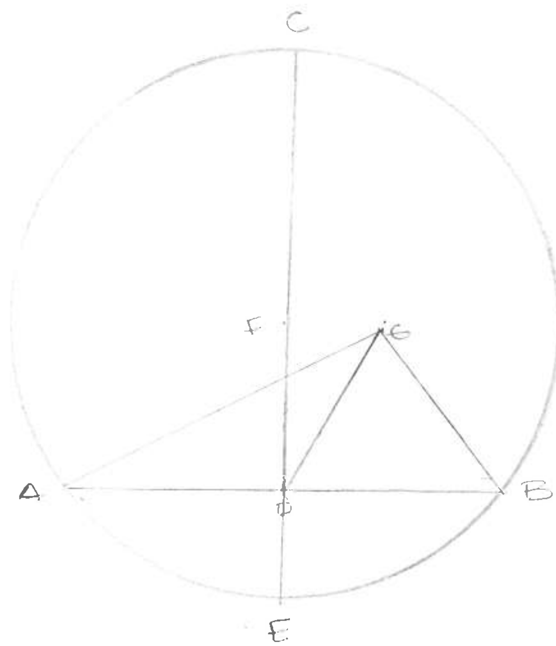
To find the centre of a given circle

Given

⊙ABC

required

find the centre of ⊙ABC



$\overline{AB}$ at random

bisect $\overline{AB}$ at D

$\overline{DC}$ at right angles to AB

DC → E

bisect $\overline{CE}$ at F

F is the centre of ⊙ABC

→ suppose G is the centre

$\overline{GA}$
$\overline{GD}$
$\overline{GB}$

AD = DB, DG is common

AD, DG = BD, DG

Base GA = Base GB
For they are radii

∡ADG = ∡GDB [I.8]

but if a straight line set up on a straight line makes the adjacent angles equal to one another, each of the angles are right. [I DEF 10]

∡GDB = ⦜

∡FDB = ⦜

∡FDB = ∡GDB
↳ impossible!

G is not the centre of ⊙ABC

Neither is any other point that is not F

F is the centre of ⊙ABC

Q E F

PORISM: FROM THIS IT IS MANIFEST THAT, IF IN A CIRCLE A STRAIGHT LINE CUT A STRAIGHT LINE INTO TWO EQUAL PARTS AND AT RIGHT ANGLES, THE CENTRE OF THE CIRCLE IS ON THE CUTTING STRAIGHT LINE

Proposition 2

If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle.

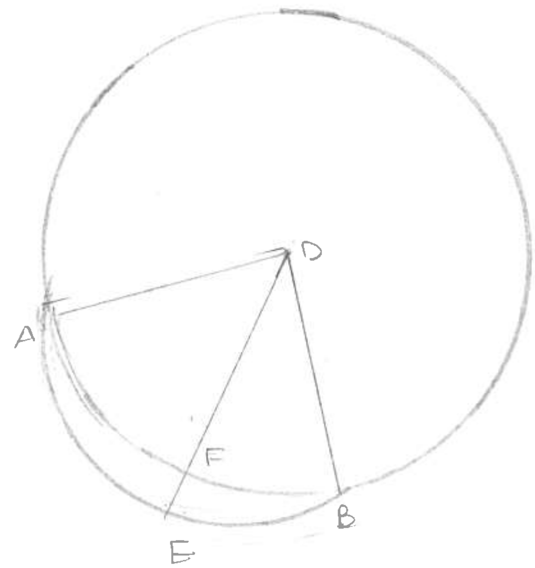
given:

⊙ABC

·A, ·B be taken at random on its circumference.

required:

$\overline{AB}$ will fall within the circle.



suppose it falls outside as AEB

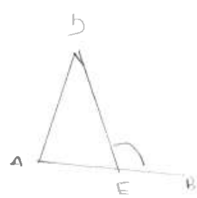
take centre D : [III.1]

$\overline{DA}$, $\overline{DB}$

$\overline{DFE}$

DA = DB

∡DAE = ∡DBE [I.5]



∡DEB > ∡DAE [I.16]

∴ For side AEB was produced

∡DAE = ∡DBE

∡DEB > ∡DBE

DB > DE [I.19]

greater angle subtends greater side

DB = DF

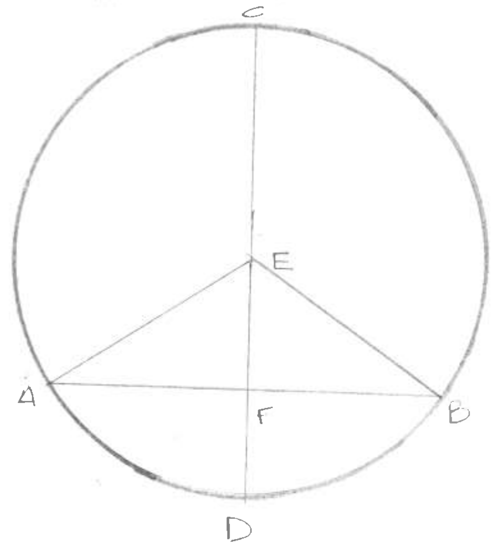
DF > DE

Impossible!

Therefore straight line AB will not fall outside the circle, but within. Q.E.D.

Proposition 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.



given:

$\bigcirc ABC$

$\overline{CD}$ cut through the centre and bisect $\overline{AB}$ at Point F.

$\overline{CD}$ cuts $\overline{AB}$ at right angles.

centre = E

$\overline{EA}$, $\overline{BE}$

AF = BF, EF common

2 sides = 2 sides

Base EA = Base BE

$\measuredangle AFE = \measuredangle BFE$ [I.8]

When a straight line, set up on a straight line makes adjacent angles equal,

The angles are right [I.DEF.10]

$\measuredangle AFE = \llcorner$

$\measuredangle BFE = \llcorner$

$\overline{CD}$ bisects $\overline{AB}$ not at the centre, at right angles.

since $\overline{CD}$ cuts $\overline{AB}$ at right angles, it bisects it

AF = BF

EA = EB

$\measuredangle EAF = \measuredangle EBF$ [I.5]

$\measuredangle AFE = \measuredangle BFE$

$\therefore$ $\triangle EAF$, $\triangle EBF$ have two angles equal to two angles and one side equal to one side (EF), and subtends one of the equal angles,

The [illegible] sides [illegible] sides AF = FB [I.26]

Q.E.D

Proposition 4

If in a circle two straight lines cut one another which are not through the centre, they do not bisect each other

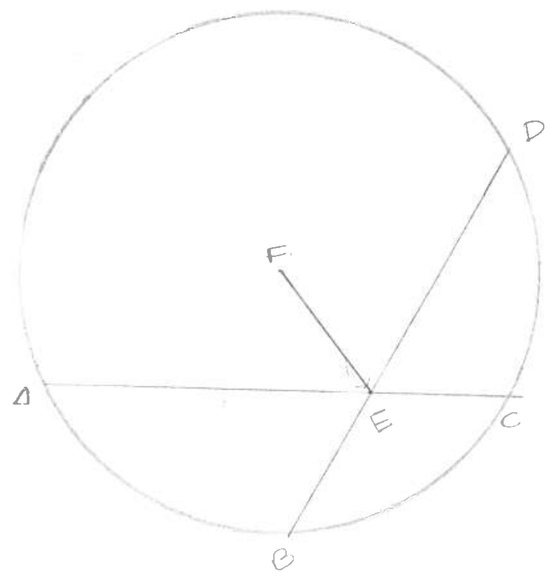
given:

⊙ABCD

$\overline{AC}$, $\overline{BD}$ cut one another at E

required:

$\overline{AC}$, $\overline{BD}$ do not bisect one another



suppose $\overline{AC}$ bisects $\overline{BD}$

AE = EC

BE = ED

centre = F [III.1]

$\overline{FE}$

FE bisects AC not through the centre

↳ it cuts it at ⊾ [III.3]

∡FEA = ⊾

FE bisects BD

↳ it cuts it at ⊾ [III.3]

∡FEB = ⊾

∡FEA = ∡FEB

∴ IMPOSSIBLE

$\overline{AC}$, $\overline{BD}$ do not bisect each other

Q.E.D.

Proposition 5

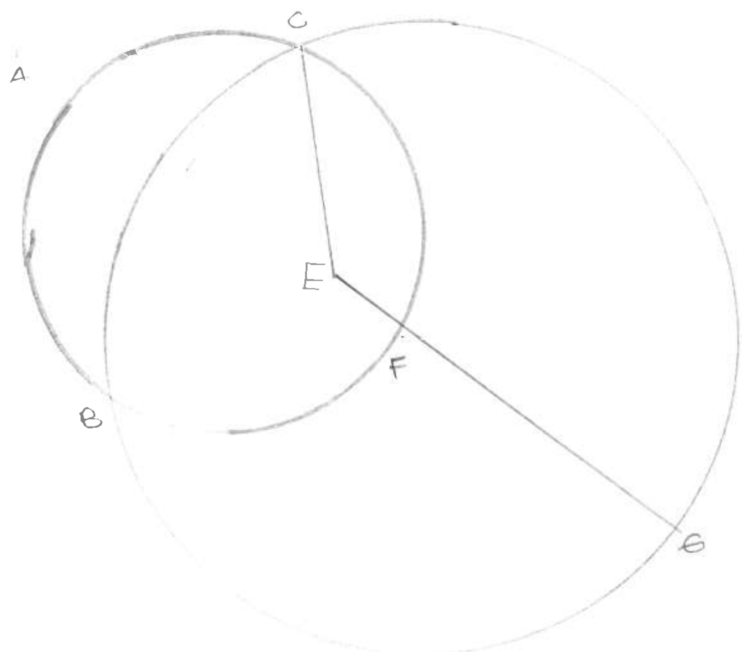
If two circles cut one another, they will not have the same centre.

given.

○ABC
○CDG
cut one another
at B, C.

required

○ABC
○CDG will
not have
the same
centre



Suppose E is the
same centre

$\overline{EC}$
$\overline{EFG}$ (at random)

E is centre of ○ABC
EC = EF [I def 15]

E is centre of CDG
EC = EG

EF = EG
IMPOSSIBLE!

E is not at the centre of
○ABC, ○DCG

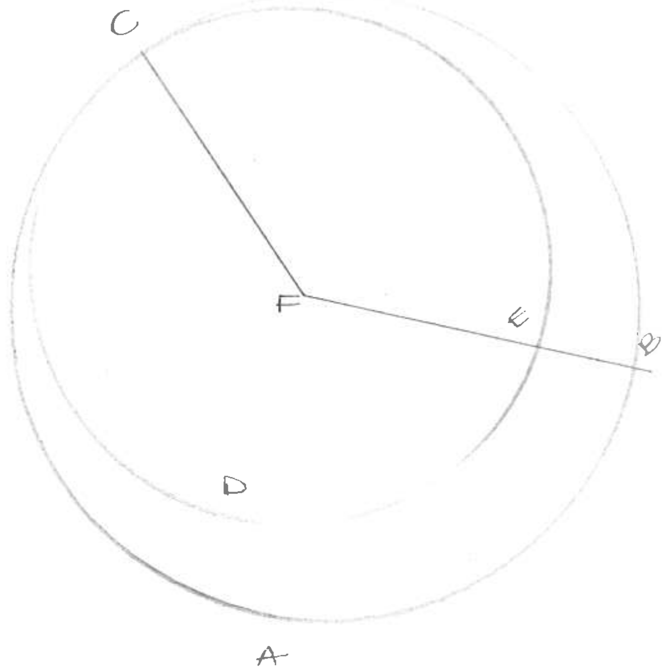
Q.E.D.

Proposition 6

If two circles touch one another, they will not have the same centre.

given:
○ABC
○CDE
Touch each other at C

required
○ABC
○CDE will not have the same centre



suppose F is the same centre
$\overline{FC}$
$\overline{FEB}$ (at random)

F is centre of ○ABC
FC = FB

F is centre of ○CDE
FC = FE

FE = FB

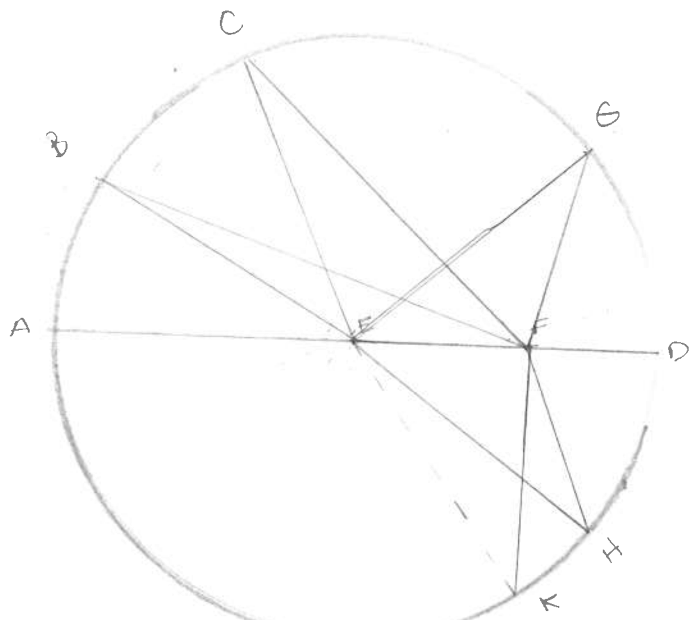
IMPOSSIBLE

F is not the centre of ○ABC, ○CDE

Q.E.D.

Proposition 7

If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle, that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest the nearer to the straight line through the centre is always greater than the more remote, and only two equal straight lines will fall from the point on the circle, one on each side of the least straight line.



Given

O ABCD
$\overline{AD}$ diameter
·F (not on centre)
E centre of circle
$\overline{FB}, \overline{FC}, \overline{FG}$

required

FA greastest, FD least
FB > FC
FC > FG
· F, only 2 equal lines will fall, one on each side of FD
→ $\overline{BE}, \overline{CE}, \overline{GE}$

EB, EF > BF
in any triangle 2 sides are greater than the remaining one [I 20]

AE = BE
AF > BF

BE = CE
FE is common

BE, EF = CE, EF
∡BEF > ∡CEF
Base BF > CF [I.24]

For the same reason
CF > FG

GF, EF > EG
EG = ED
GF, EF > ED
→ subtract EF
GF > FD

FA is greatest, FD is least
FB > FC ∴ FC > FG

→ on EF & E
∡FEH
∡FEH = ∡GEF [I.23]
$\overline{FH}$

GE = EH
EF is common

GE, EF = HE, EF
∡GEF = ∡HEF
base FG = base FH [I.4]

suppose

a straight line = $\overline{FG}$ will not fall on the circle from ·F

$\overline{FK}$

FK = FG
FH = FG
FK = FH

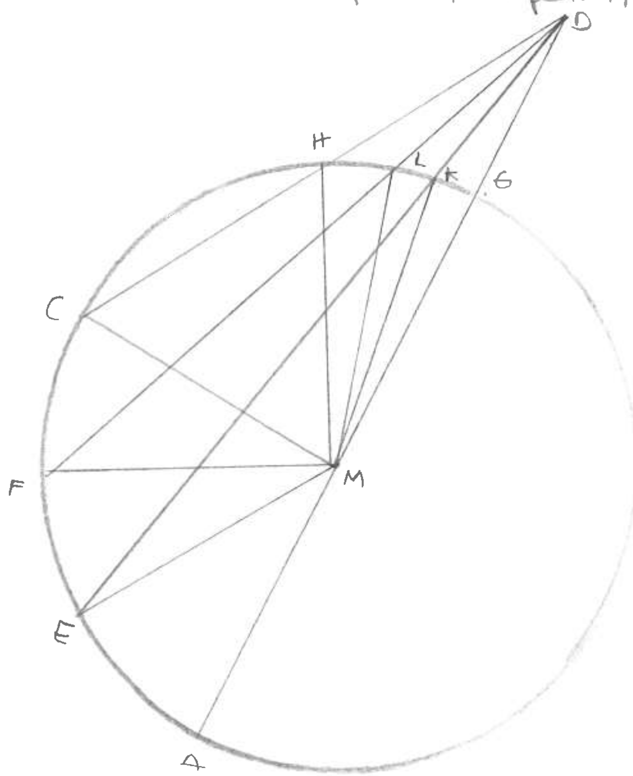
impossible

↳ no other line = GF will fall on the circle

QED

Proposition 8

If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least.



given:

⊙ABC
·D
$\overline{DA}, \overline{DE}, \overline{DF}, \overline{DC}$
$\overline{DA}$ through the centre

required:

on concave circumference AEFC
$\overline{DA}$ is the greatest
DE > DF, DF > DC
on convex circumference HLKG
$\overline{DG}$ is le least
nearer to the least is less
DK < DL, DL < DH
2 = straight lines fall from D one on each side of DG

→ centre M [III.1]
$\overline{ME}, \overline{MF}, \overline{MC}, \overline{MK}, \overline{ML}, \overline{MH}$

AM = EM
→ add MD
AD = EM, MD

EM, MD > ED
AD = ED

ME = MF
MD is common
EM, MD = MF, MD
∡EMD > ∡FMD
Base ED > FD [I.24]

Similarly,
FD > CD

DA is the greatest
DE > FD, FD > CD

MK, KD > MD
MG = MK
→ subtract MK, MG
KD > GD

MK, KD < ML, LD [I.21]
for MK, KD were constructed within triangle MLD

MK = ML
→ subtract MK, ML
KD < LD

Similarly,
DL < DH

DG is the least
DK < DL, DL < DH

on MD, ·M
∡DMB
∡DMB = ∡KMD
$\overline{DB}$

MK = MB
MD is common
KM, MD = BM, MD
∡KMD = ∡BMD
Base DK = DB [I.4]

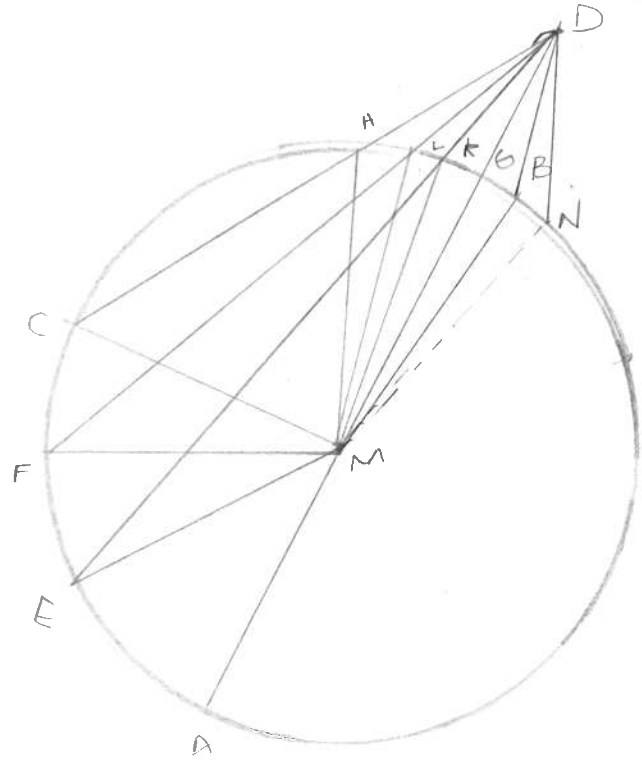
→ no other straight line = DK will fall on the circle from ·D

suppose

DN = DK
DB = DK
DN = DB

→ The nearer to the least DG = to the more remote
(Farther away the greater)
IMPOSSIBLE

No two straight lines will fall on ⊙ABC from ·D



QED

Proposition 9

If a point be taken within a circle, and more than two equal straight lines fall from the point, the point taken is the centre of the circle.

given:

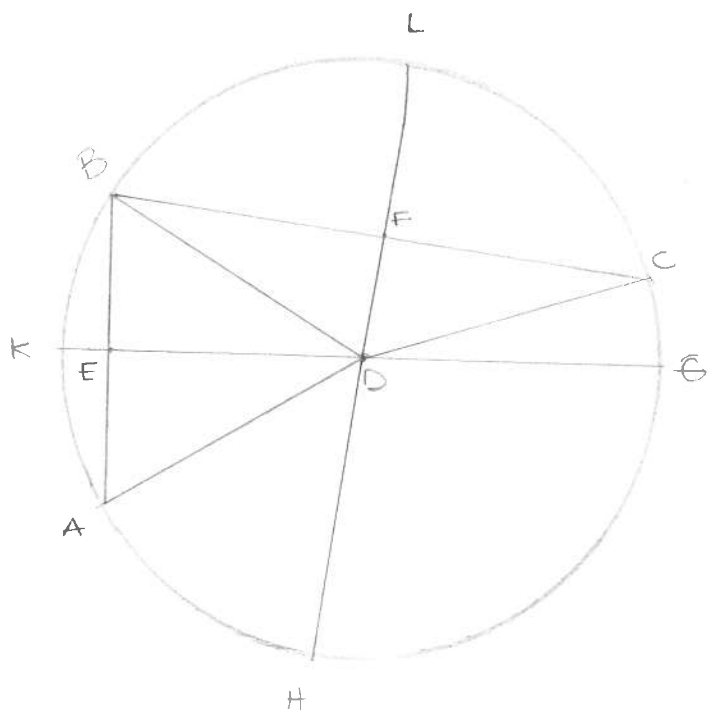
⊙ABC

·D

$\overline{DA} = \overline{DB}, \overline{DC}$

required:

D is centre of ⊙ABC



$\overline{AB}, \overline{BC}$ bisected at E, F

$\overline{ED}, \overline{FD}$ drawn through to G, H, K, L

AE = BE
ED is common

AE, ED = BE, ED
base DA = base DB
∡AED = ∡BED [I. 8]

∡AED, ∡BED = 2∟ [I. Def. 10]
GK cuts AB into equal parts and equal angles

If in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line [III. 1, porism]

centre of circle is GK.

For the same reason,
centre of circle is in HL

The only point HL, GK have in common is D

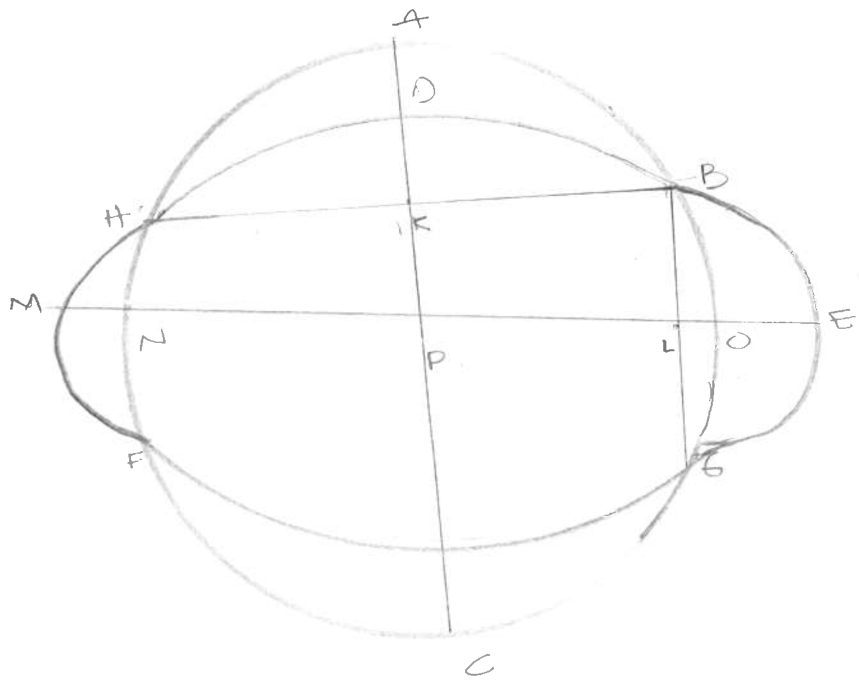
D is the centre of ⊙ABC Q.E.D.

Proposition 10

A circle does not cut a circle at more points than two

given:

OABC cut ODEF at more points than 2 B, G, F, H



required:

O ABC does not cut O DEF at more points than 2

$\overline{BH}, \overline{BG}$ bisected at K, L
$\overline{KC}, \overline{LM}$ at rt. to $\overline{BH}, \overline{BG}$ & carried through A, E

in O ABC
AC cuts BH into 2 equal parts and at rt.
AC is the centre [III. 1, Por]

in O ABC
NO cuts BG into equal parts and at rt.
NO is the centre

AC, NO only meet at P
P is the centre of OABC

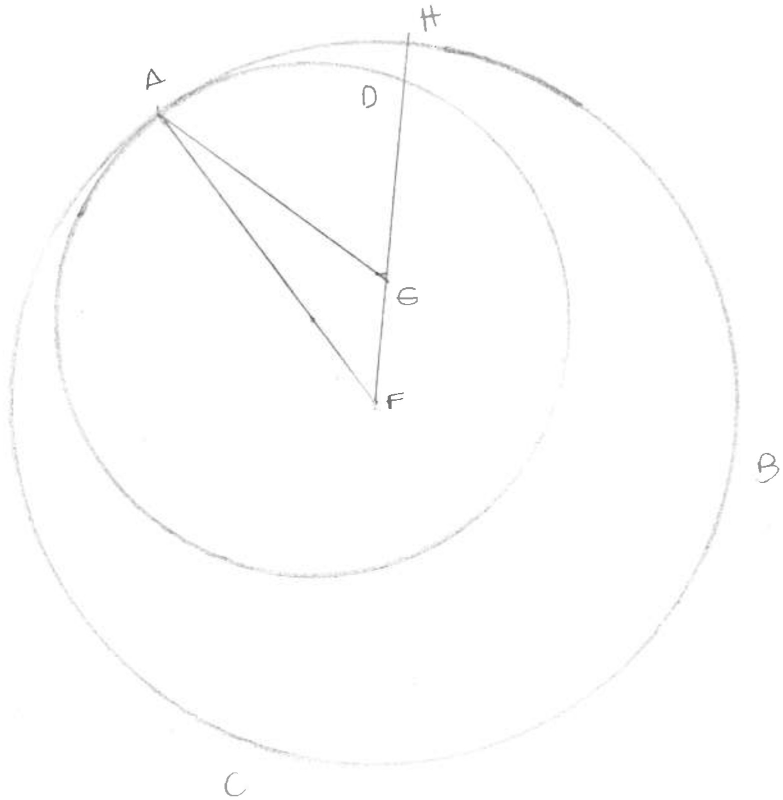
Similarly,
P is the centre of O DEF

OABC, ODEF have the same centre P,
IMPOSSIBLE [III.5]

QED

Proposition 11

If two circles touch one another internally, and their centres be taken, the straight line joining their centres, if it also be produced, will fall on the point of contact of the circles.



given:

OABC, OADE
Touch each other at A
·F centre OABC
·G centre OADE

required:

$\overline{GF}$ produced will fall on A

suppose,
The uniting of centres falls on FGH

$\overline{AF}$, $\overline{AG}$

AG, GF > FA
FA = FH
AG, GF > FH
→ Subtract FG
AG > GH

AG = GD
GD > GH
IMPOSSIBLE

Straight line joined from F to G will fall on A, the point of contact.

Q.E.D.

Proposition 12

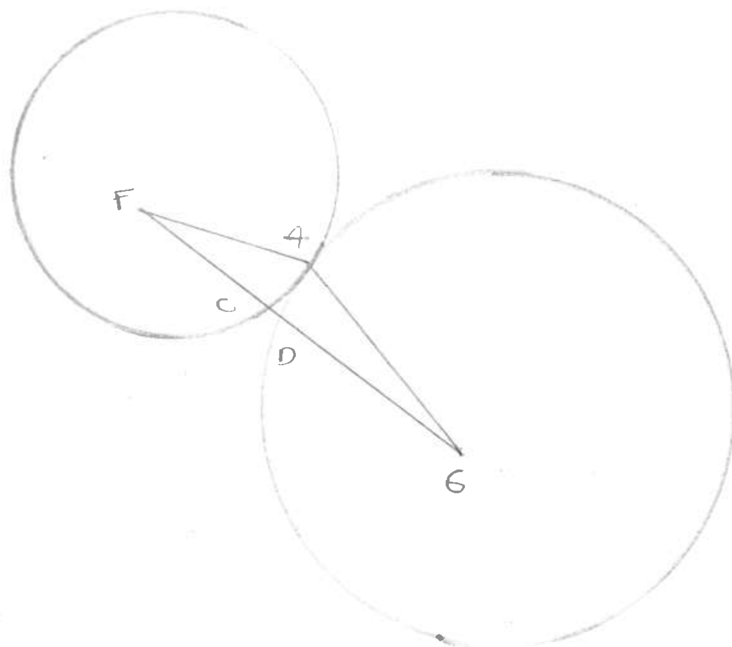
If two circles touch each other externally, the straight line joining their centres will pass through the point of contact.

given:

○ABC
○ADE
touch one another at A
F centre ○ABC
G centre ○ADE

required:

Straight line from F to G will pass through A



Suppose,
The straight line will pass as GDCF
$\overline{AF}$, $\overline{AG}$

○ABC
FA = FC

○ADE
GD = GA

FA, AG = FC, DG

FG > FA, AG

but it is also less [I.20]
IMPOSSIBLE

Straight line joining F & G will not fail to pass through A

Q.E.D.

Proposition 13

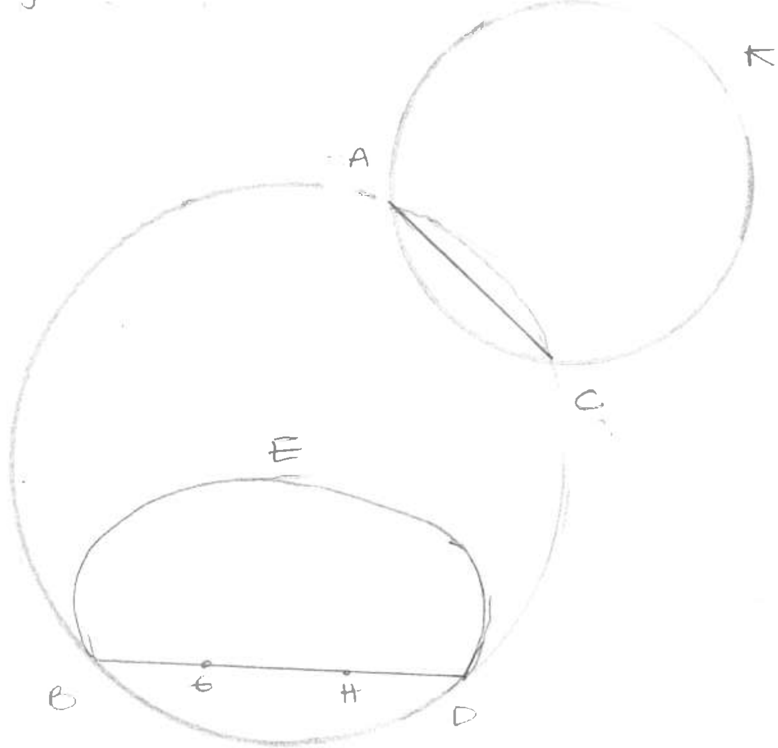
A circle does not touch a circle at more points than one, whether it touch it externally or internally.

Given:

○ABCD touch
○EBFD (internally)
at more than one point,
D, B.

Required:

○ABCD touches
○EBFD at only one
point.
(internally or
externally)



G centre of ○ABCD
H centre of ○EBFD

GH

GH will fall on B, D [III.11]
BGHD

○ABCD
BG = GD

BG > HD

BH >> HD

○EBFD
BH = HD

IMPOSSIBLE

internally a circle
does not touch
another circle at
more points
than 1.

Given:

○ACK touch ○ABDC
at A, C.

Required

○ACK touches ○ABDC
once.

AC

Since on the circumference of ○ABCD, ○ACK
two points taken at random, the straight line
joining the points will fall within each circle.
[III.2]

It fell on ○ABCD, outside of ○ACK. [III. Def. 3]

ABSURD

externally a circle does not touch
another circle at more points than 1.

QED.

Proposition 14

In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another

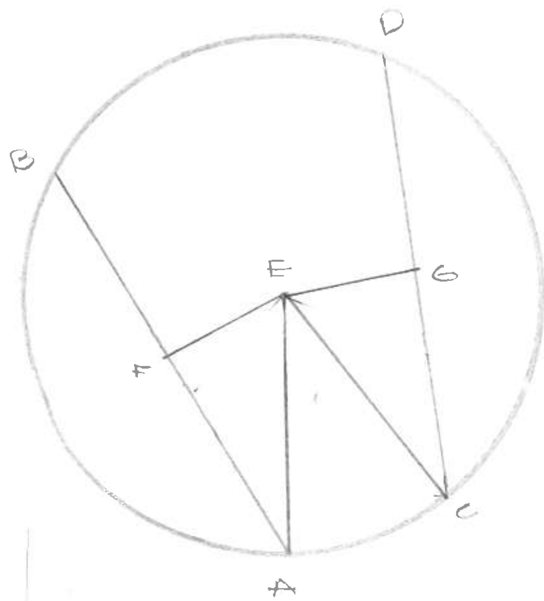
given.

○ABDC
$\overline{AB} = \overline{CD}$

required.

$\overline{AB}, \overline{CD}$ are equally distant from the centre.

$\overline{AB} = \overline{CD}$



centre E of ○ABDC [III.1]

$\overline{EF}, \overline{EG}$ Perpendicular to AB, CD

$\overline{AE}, \overline{EC}$

EF cuts AB at right angles and bisects it [III.3]

AF = BF
AB is double AF

For the same reason:
CD is double CG.

AB = CD

CG = AF

AE = EC
square on AE = square on EC

squares on AF, EF = square on AE
∡F = ⊾ [I.47]

squares on EG, GC = square on EC
∡G = ⊾ [I.47]

squares AF, EF = squares EG, GC
square AF = square GC
AF = GC
square EF = square EG
EF = GC

In a circle, straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal [III. def 4]

AB, CD are equally distant from the centre.

→ AB, CD equally distant from centre
AB is double AF
CD is double CG

AE = CE
square on AE = square on CE
Squares on EF, FA = square EA [I.47]
Squares EG, GC = square CE

squares EF, FA = squares EG, GC
square EF = square EG
square FA = square GC.

AF = GC

AB is double AF
CD is double GC

AB = CD

Q.E.D

Proposition 15

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

<u>given:</u>

$\odot ABCD$

$\overline{AD}$ diameter

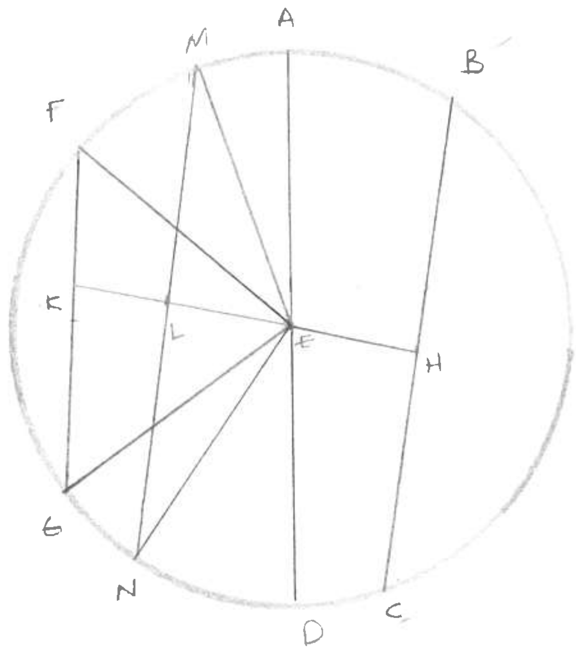
E centre

$\overline{BC}$ nearer to $\overline{AD}$

$\overline{FG}$ more remote

<u>required:</u>

$AD > BC > FG$



$\overline{EH}, \overline{HK}$ perpendicular to BC, FG

$EK > EH$

BC is nearer to the centre

FG more remote [III def. 5]

$\overline{EL}$

$EL = HE$

$\overline{LM}, \overline{MN}$

at ⊥ EK

$\overline{ME}, \overline{EN}, \overline{FE}, \overline{EG}$

$EH = EL$

$BC = MN$ [III.14]

$AE = EM$

$ED = EN$

$AD = EM, EN$

$ME, EN > MN$ [I.20]

$MN = BC$

$AD > BC$

$ME, EN = FE, EG$

$\measuredangle MEN > \measuredangle FEG$

base $MN >$ base FG [I.24]

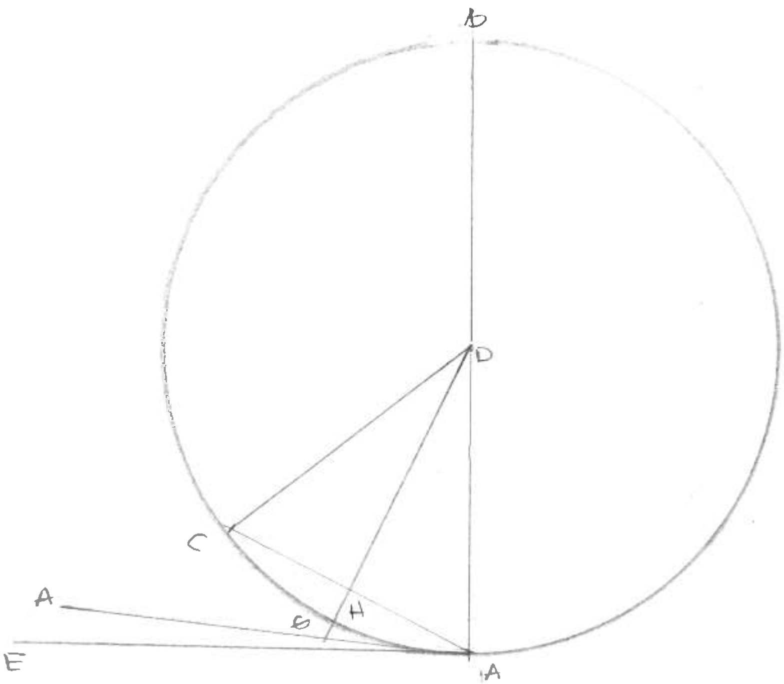
$MN = BC$

AD is greatest, $BC > FG$

Q.E.D.

Proposition 16

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.



given:

○ABC
D centre
AB diameter

required

straight line drawn from A at right angles with $\overline{AB}$ will fall outside the circle

suppose:
The straight line from A will fall within as CA

$\overline{DC}$

DA = DC

∡DAC = ∡ACD [I.5]

∡DAC = ⊾

∡ACD = ⊾

in △ACD
∡DAC, ∡ACD = 2⊾!
IMPOSSIBLE [I.17]

↳ The straight line drawn from A at ⊾ to BA will not fall within the circle

Similarly, we can prove it will not fall on the circumference.

Therefore, it will fall outside.

→ let it fall as AE

required

in the space between AE and the circumference CHA another line cannot be interposed

suppose:
another line interposes as $\overline{FA}$

$\overline{DG}$ perpendicular to $\overline{FA}$

∡AGD = ⊾

∡DAG < ⊾

AD > DG [I.19]

DA = DH

DH > DG
IMPOSSIBLE

↳ A straight line cannot be interposed into the space between the straight line and the circumference.

required:

$\angle$ of the semicircle contained by $\overline{BA}$ and circumference CHA
$>$ any acute rectilineal angle

remaining angle contained by CHA and line AE $<$ any acute rectilineal angle

IF There is a rectlineal $\angle$ $>$ angle contained by $\overline{BA}$ and CHA and any rectilineal $\angle$ $<$ angle contained by CHA and $\overline{AE}$

Then a straight line will be interposed and will make $\angle$ contained by straight lines which is greater than $\angle$ Contained by BA and CHA,
and another $\angle$ contained by straight lines $<$ $\angle$ contained by CHA ang AE

IMPOSSIBLE

a straight line cannot be interposed
↳ There will be no acute $\angle$ contained by straight lines $>$ than $\angle$ contained by BA and circumference

nor

acute $\angle$ contained by straight lines $<$ $\angle$ contained by CHA and AE

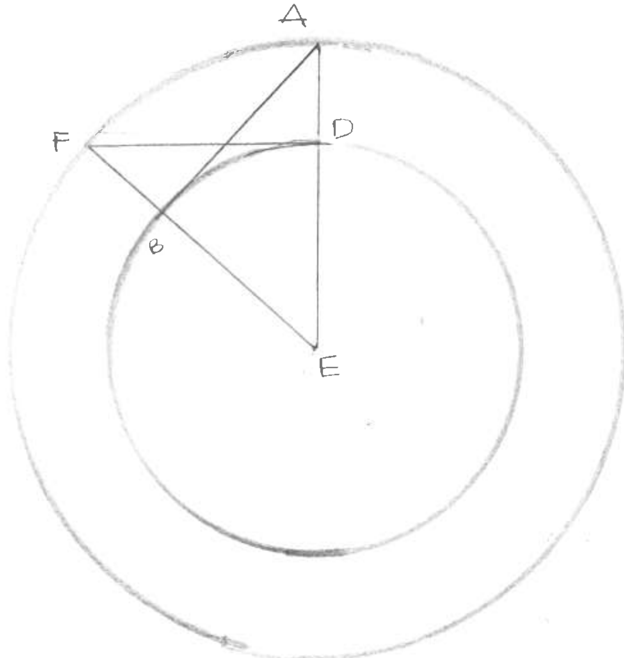
Q.E.D.

PORISM:

It is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

Proposition 17

From a given point to draw a straight line touching a given circle.



given:

A
⊙BCD

required:

from ·A a straight line touching ⊙BCD

centre E [III.1]

$\overline{AE}$

⊙AFG
 centre E, distance $\overline{AE}$

$\overline{DF}$
 at ⊾ to EA

$\overline{EF}$, $\overline{AB}$

→ AB has been drawn from ·A touching ⊙ABD

E is centre of ⊙BCD, ⊙AFG
 EA = EF
 ED = EB

EA, EB = EF, ED
 ∡E is common

base DF = base AB

△DEF = △BEA

remaining ∡ = remaining ∡ [I.4]

∡EDF = ∡EBA

∡EDF = ⊾

∡EBA = ⊾

EB is a radius
 a straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle.

AB touches ⊙BCD

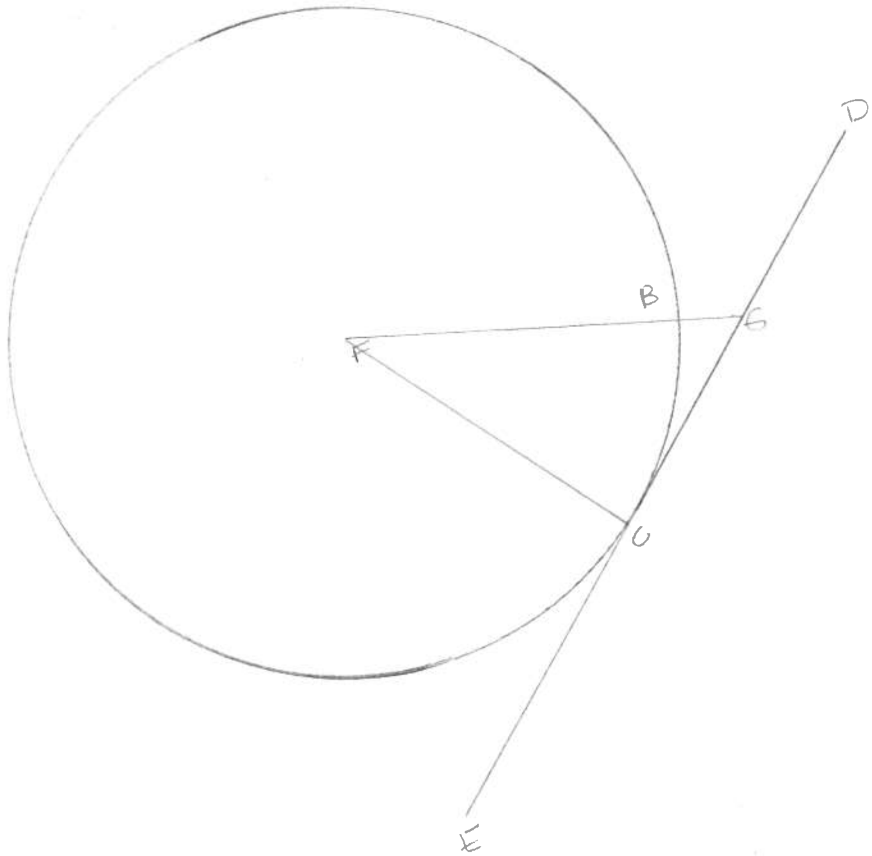
given point A, a line $\overline{AB}$ touches ⊙BCD

Q.E.F.

❤ Proposition 18

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

given:
$\overline{DE}$ touch
⊙ABC at C.
centre F
$\overline{FC}$
required:
FC is perpendicular to DE



Suppose:
$\overline{FG}$ is perpendicular to $\overline{DE}$

∡FGC = ⊾
∡FCG is acute [I.17]

FC > FG [I.19]
greater angle is subtended by greater side

FC = FB

FB > FG

IMPOSSIBLE

FG is not perpendicular to DE

Similarly,
neither is any other side except FC

FC is perpendicular to DE.

QED.

Proposition 19

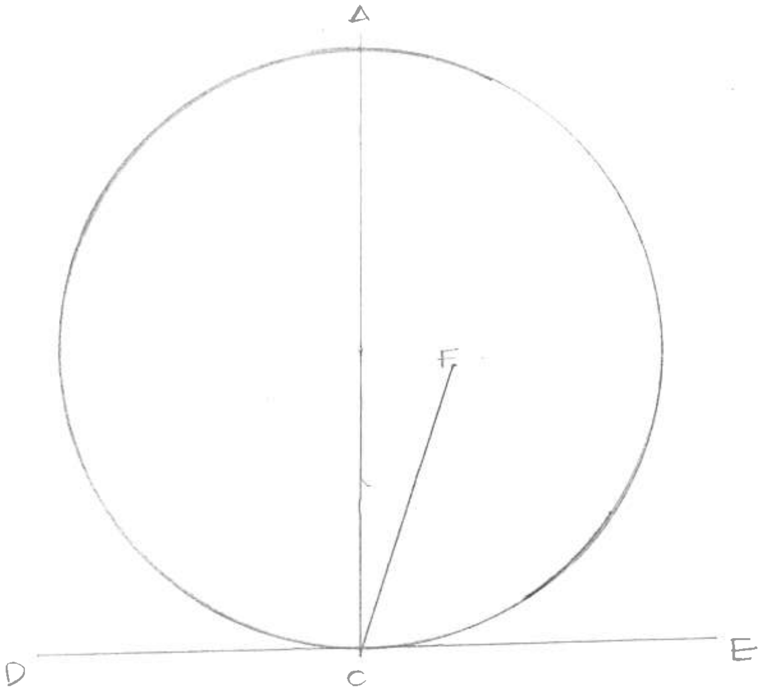
If a straight line touch a circle, and from the point of contact a straight line will be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

given:

DE touch
OABC at C
CA be drawn at
⊥ to DE

required

centre of
O is on AC



suppose:
F is centre
CF

DE touches OABC
FC has been joined from the centre to point of contact

FC is perpendicular to DE
∡FCE = ∟ [III.18]

∡ACE = ∟

∡FCE = ∡ACE
IMPOSSIBLE

F is not the centre of OABC

similarly,
neither is any point expect on AC.

Q E D.

Proposition 20

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

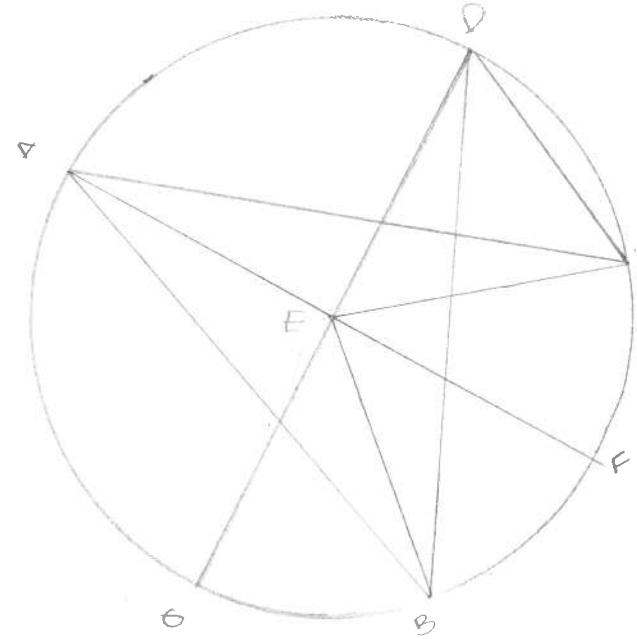
given:

⊙ABC

∡BEC at its centre

∡BAC at its circumference

same circumference BC as base



required

∡BEC is double ∡BAC

$\overline{AE} \to F$

EA = EB

∡EAB = ∡EBA [I.5]

∡EAB, ∡EBA = 2 ∡EAB

∡BEF = ∡EAB, ∡EBA [I.32]

∡BEF = 2 ∡EAB

For the same reason,

∡FEC = 2 ∡EAC

∡BEC = 2 ∡BAC [2∡EAC, 2 ∡EAB]

→ ∡BDC

$\overline{DE} \to G$

Similarly,

∡GEC = 2 ∡EDC

∡GEB = 2 ∡EDB

∡BEC = 2 ∡BDC

Q.E.D.

Proposition 21

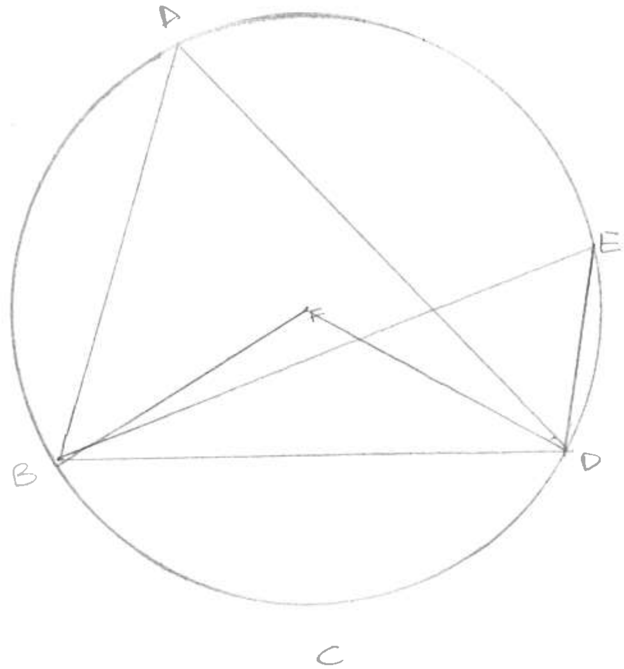
In a circle the angles in the same segment are equal to one another

given:

⊙ABCD
∡BAD } angles in the same segment
∡BED } BAED

required

∡BAD = ∡BED



centre F
$\overline{BF}$, $\overline{FD}$

∡BFD is at centre
∡BAD at circumference
same circumference BCD as base
∡BFD is double ∡BAD [III.20]

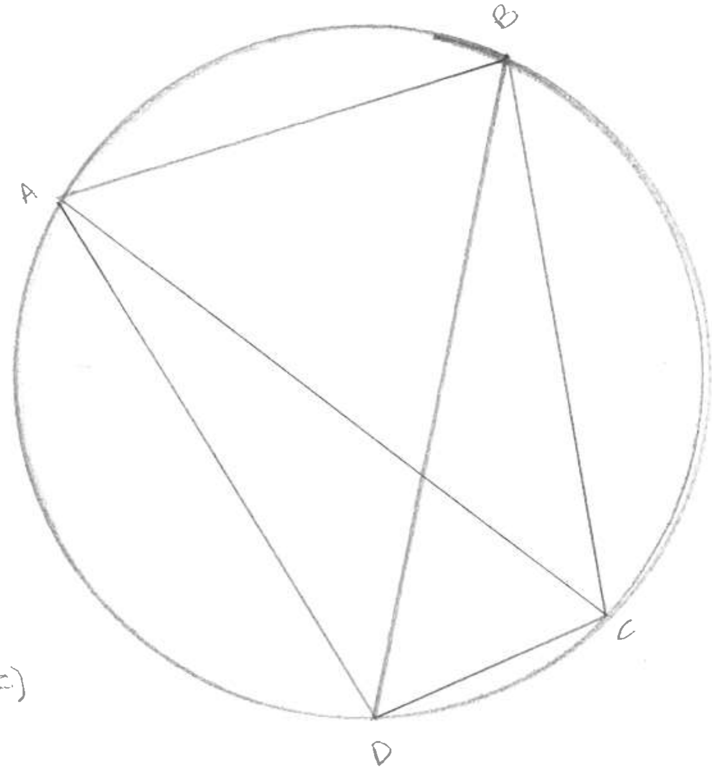
For the same reason,
∡BFD is double ∡BED

∡BAD = ∡BED

QED.

Proposition 22

The opposite angles of quadrilaterals in circles are equal to two right angles



given.

○ABCD

□ABCD

required

opposite angles = 2L

$\overline{AC}$, $\overline{BD}$

In any triangle, the 3 angles = 2L [I.32]

∡CAB, ∡ABC, ∡BCA = 2L (△ABC)

∡CAB = ∡BDC [III.21]
they are in the same segment BADC

∡ACB = ∡ADB
they are in the same segment ADCB

∡ADC = ∡ADB, ∡BDC

∡ADC = ∡CAB, ∡ACB

→ add ∡ABC

∡ADC, ∡ABC = ∡CAB, ∡ACB, ∡ABC

∡CAB, ∡ABC, ∡BCA = 2L

∡ADC, ∡ABC = 2L

similarly,

∡BAD, ∡DCB = 2L

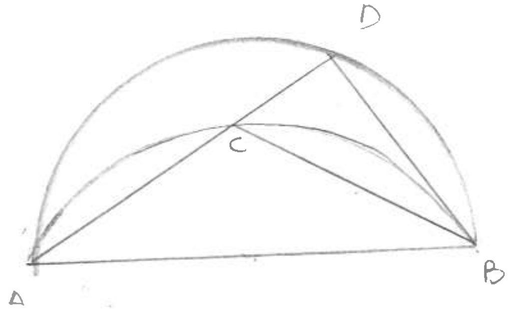
Q.E.D

Proposition 23

on the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

given/required:

AB have
o similar
nequal segments
f O ACB.



ACD
CB, DB

segment ACB is similar to segment ADB

similar segments of circles are those which admit equal angles [III.def 11]

$\measuredangle ACB = \measuredangle ADB$

nterior to exterior; [I.16]

IMPOSSIBLE

Q.E.D.

Proposition 24

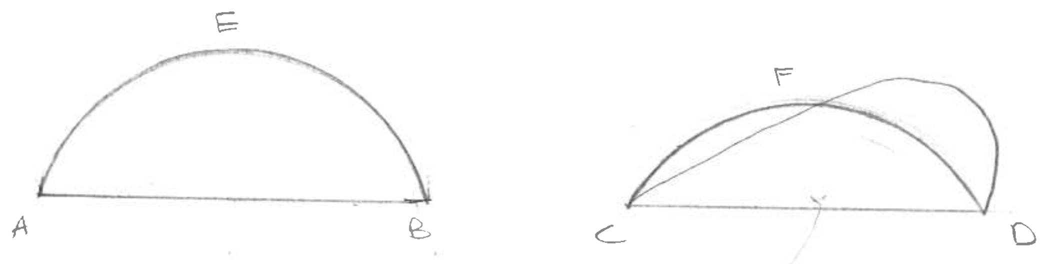
Similar segments of circles on equal straight lines are equal to one another

given

AEB, CFD be similar segments of circles on equal lines $\overline{AB}$, $\overline{CD}$.

required

segment AEB = segment CFD



IF segment AEB be applied to CFD
A will be on C
$\overline{AB}$ will be on $\overline{CD}$
B willbe on D

since AB = CD and they coincide

segment AEB will coincide with CFD

IF AB coincides with CD, but
segment AEB doesn't coincide with CFD

it will fall within or outside it.
or awry like CGD

CGD cuts the circle at more parts than two.

IMPOSSIBLE [III.10]

→IF $\overline{AB}$ be applied to $\overline{CD}$
AEB will coincide with CFD
and will be equal

Q.E.D.

Proposition 25

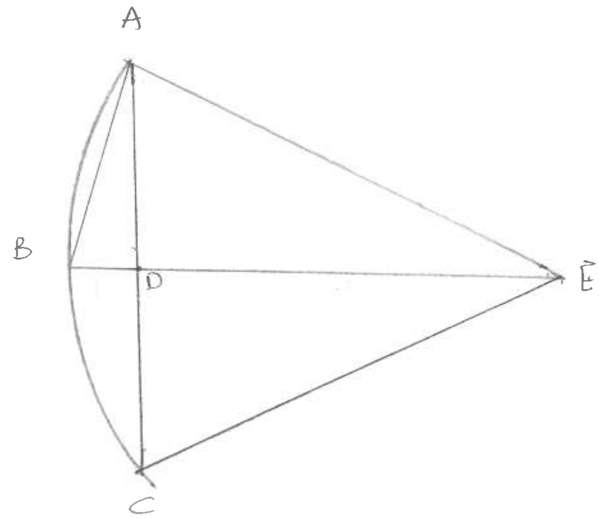
Given a segment of a circle, to describe the complete circle of which it is a segment.

given:

ABC segment of O

required:

describe the complete circle belonging to ABC



bisect AC at D

$\overline{DB}$ at right ∡ with AC

$\overline{AB}$

∡ADB is either >, =, < than ∡BAD

→ let ∡ADB > ∡BAD

∡BAE be constructed = ∡ABD

$\overline{FB}$ → E

$\overline{EC}$

∡ABE = ∡BAE

$\overline{EB}$ = $\overline{EA}$ [I.6]

$\overline{AD}$ = $\overline{DC}$

DE common

AD, DE = DC, DE

∡ADE = ∡CDE

base ΔE = base CE

AE = BE

BE = EC

AE = BE = EC

Centre is E

AE, EB, EC will also pass the remaining points and will have been completed [III.9]

The completed circle has been described

→ ABC < semicircle

E is outside of ABC

Similarly,

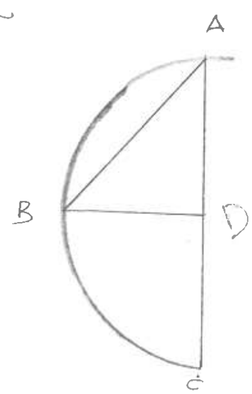
if ∡ABD = ∡BAD

AD = BD, DC

DA = BD = DC

D will be the centre

ABC will be a semicircle



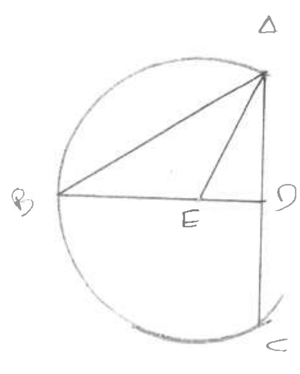
→ ∡ABD < ∡BAD

if on BA we construct

∡ = ∡ABD

centre will be on DB within ADB

ABC > semicircle



Q.E.F

Proposition 26

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

given

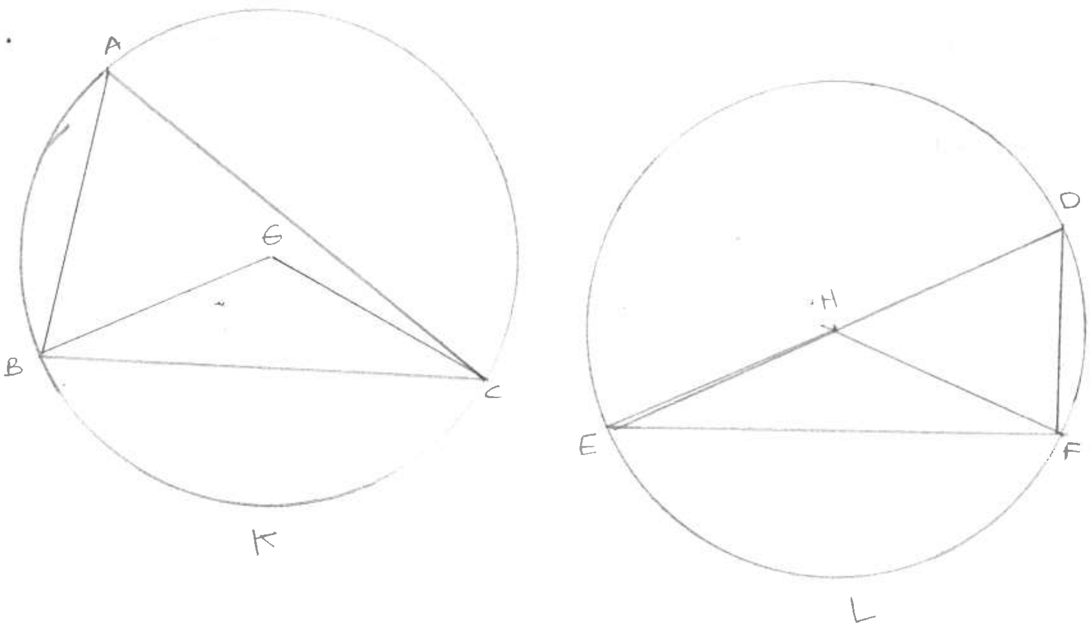
⊙ABC = ⊙DEF

∡BGC = ∡EHF (centres)

∡BAC = ∡EDF (circum)

required

circumference BKC =
circumference ELF



$\overline{BC}$
$\overline{EF}$

⊙ABC = ⊙DEF
radii are equal
BG, GC = EH, HF
∡G = ∡H
base BC = base EF [I.4]

∡A = ∡D
BAC is similar to EDF [III. def 11]
They are upon straight lines.

Similar segments on equal straight lines are equal [III.24]
BAC = EDF

⊙ABC = ⊙DFE
circumference BKC = circumference ELF

QED

Proposition 27

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

given:

$\odot ABC = \odot DEF$

circumference BC = circum EF

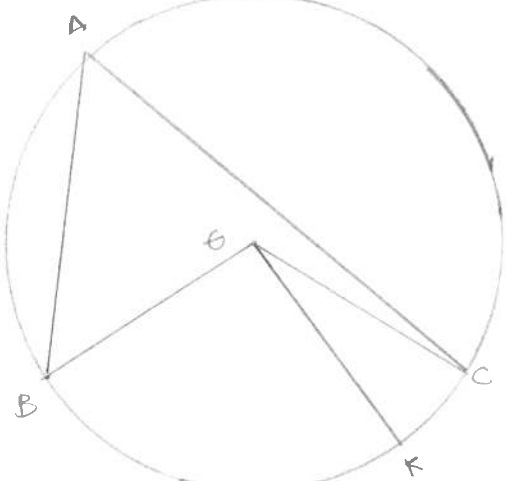
$\angle BGC$, $\angle EHF$ stand at centre.

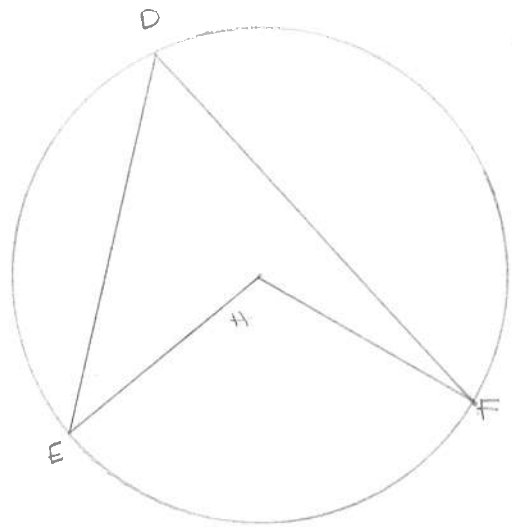
$\angle BAC$, $\angle EDF$ stand at circum

required:

$\angle BGC = \angle EHF$

$\angle BAC = \angle EDF$





suppose

$\angle BGC > \angle EHF$

on BG

$\angle BGK = \angle EHF$ [I.23]

equal angles stand on equal circumferences when they are at centres [III.26]

circumference BK = circumference EF

$EF = BC$

$BK = BC$

IMPOSSIBLE

$\angle BGC = \angle EHF$

$\angle A$ = half of $\angle BGC$ [III.20]

$\angle D$ = half of $\angle EHF$

$\angle A = \angle D$

QED

Proposition 28

In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less

given:

○ABC = ○EDF

$\overline{AB} = \overline{DE}$

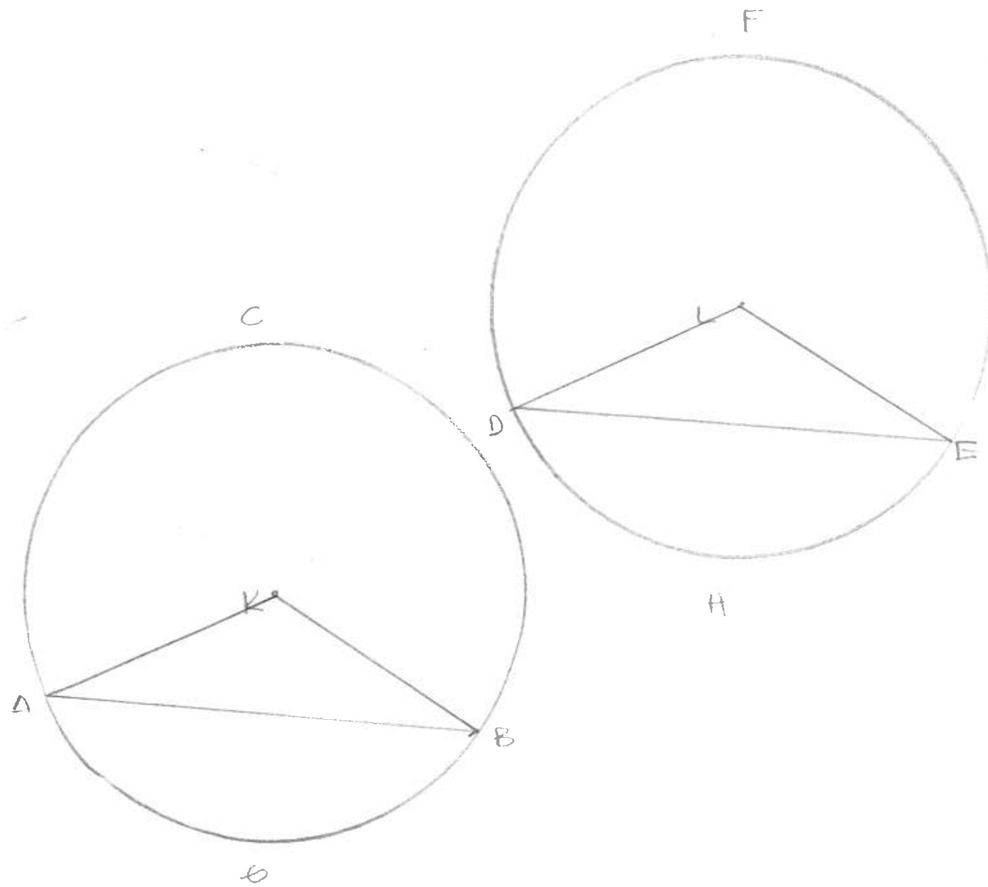
cutting off ACB, DFE as greater circumferences

AGB, DHE as lesser

required:

circum ACB = circum DFE

circum AGB = circum DHE



centres K, L

$\overline{AK}, \overline{KB}$

$\overline{DL}, \overline{LE}$

circles are equal,

radii are equal

AK, KL, = DL, LE

base AB = base DE

∡ AKB = ∡ DLE [I.8]

equal angles stand on equal circumferences, when they are at centres [III.26]

circumference AGB = circumference DHE

○ABC = ○DFE

circumferences that remain are equal

ACB = DFE

Q.E.D

Proposition 29

In equal circles equal circumferences are subtended by equal straight lines.

given

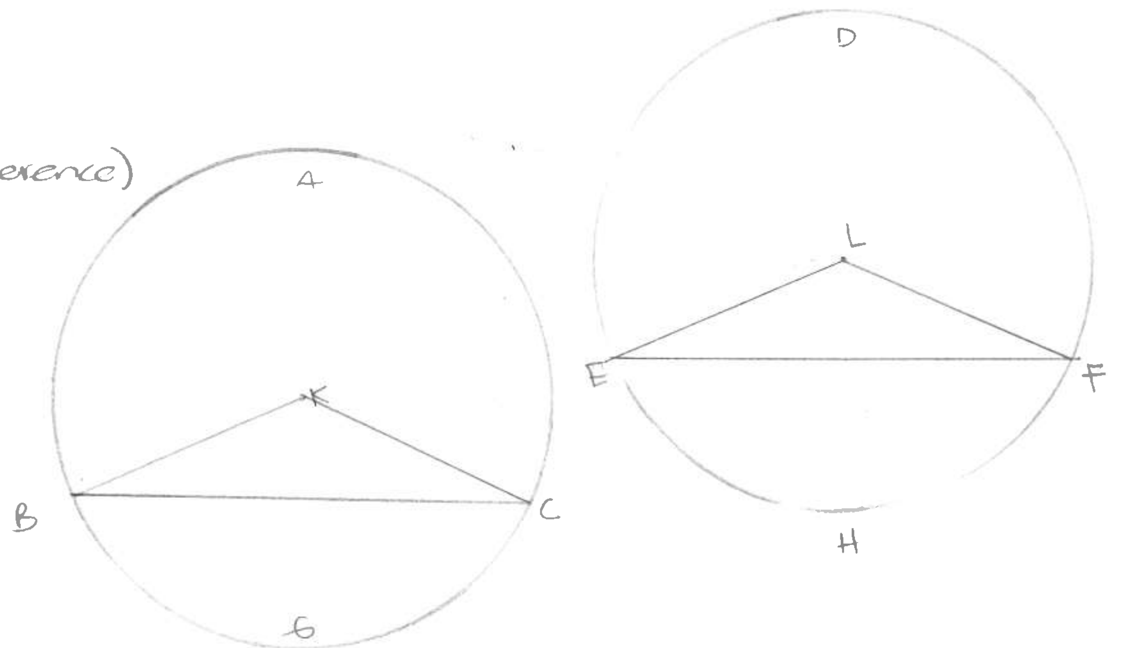
⊙ABC = ⊙DEF

BGC = EHF (circumference)

$\overline{BC}$, $\overline{EF}$

required:

BC = EF



centres K, L

$\overline{BK}$, $\overline{KL}$

$\overline{EL}$, $\overline{LF}$

circum BGC = circum EHF

∢BKC = ∢ELF [III.27]

⊙ABC = ⊙DEF

radii are equal

BK, KC = EL, FL

They have equal angles

base BC = base EF [I.4]

QED

Proposition 30

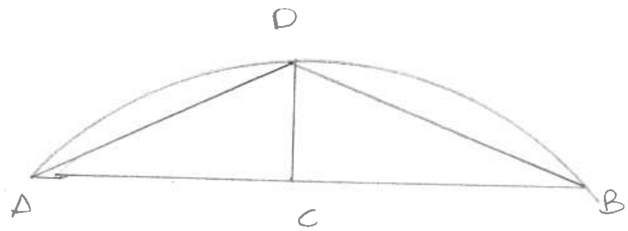
To bisect a given circumference.

given:

circumference ADB

required:

bisect ADB



$\overline{AB}$

bisect AB at C

$\overline{CD}$ at right angles to AB

$\overline{AD}$, $\overline{DB}$

AC = CB
  CD is common

AC, CD = BC, CD

∡ ACD = ∡ BCD
  each is right

base AD = base BD [I.4]

equal straight lines cut off equal circumferences, [III.28]
greater = greater less = less

circumferences AD, DB are less than a semicircle
AD = DB

Circumference ADB has been bisected

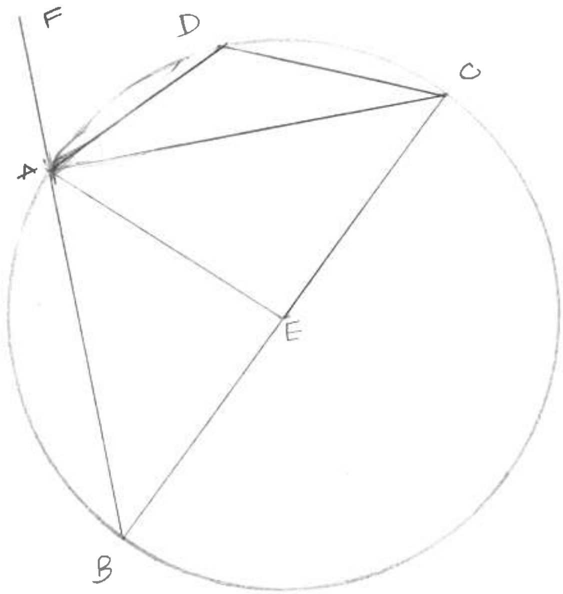
QEF

◻ = quadrilateral
◡ = semicircle
○ = circle

angle of ◻ = ∟

Proposition 31

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.



given:

◯ABCD
BC diameter
E centre
$\overline{BA}, \overline{AC}, \overline{AD}, \overline{DC}$

required:

∡BAC in semicircle BAC = ∟
∡ABC in segment ABC < ∟
(greater)
∡ADC in segment ADC > ∟
(less)

→

$\overline{AE}$
BA → F
BE = EA
∡ABE = ∡BAE [I.5]

CE = EA
∡ACE = ∡EAC [I.5]

∡BAC = ∡BAE, ∡EAC
∡BAC = ∡ABC, ∡ACB

∡FAC is exterior to △ABC
∡FAC = ∡ABC, ∡ACB

∡BAC = ∡FAC
each is right [I.Def.10]
∡BAC in semicircle BAC is right

→ in △ABC
∡ABC, ∡BAC < 2∟ [I.17]

∡BAC = ∟
∡ABC < ∟

∡ABC is the ∡ in segment ABC > ◡

◻ABCD is a ◻ in a ○
opposite ∡ of ◻ in ○ = 2∟ [III.22]

∡ABC < ∟ } opposite ∡
∡ADC > ∟ }
∡ADC is the ∡ in segment ADC < ◡

→
required:
∡ of the greater segment > ∟ (ABC)
∡ of the lesser segment < ∟ (ADC)

∡ contained by BA, AC = ∟
∡ contained by circum ABC, AC > ∟

∡ contained by AC, AF = ∟
∡ contained by CA, circum ADC < ∟

Q.E.D.

Proposition 32

If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

given

$\overline{EF}$ touch ⊙ABCD at B.

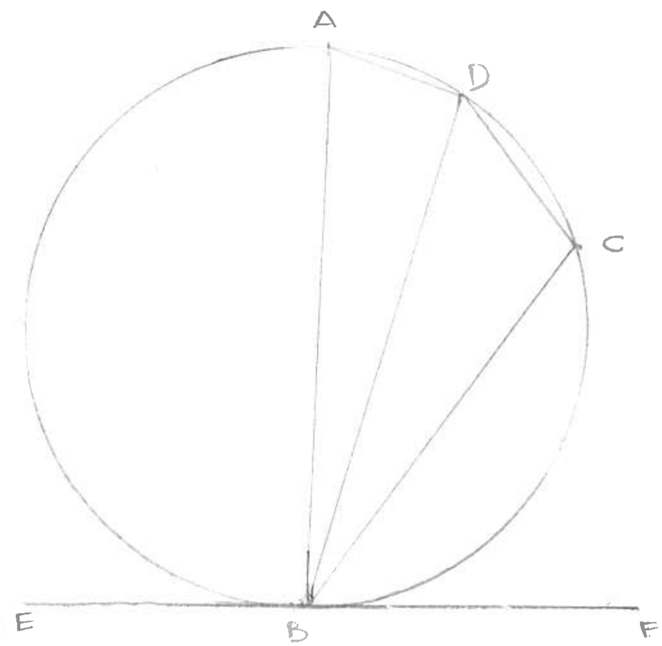
$\overline{BD}$ cutting the ⊙

required:

∡BD makes with $\overline{EF}$ = the ∡ in alternate segments

∡FBD = ∡ in segment BAD

∡EBD = ∡ in segment DCB



$\overline{BA}$ at ⊥ with $\overline{EF}$

C at random on circum BD

$\overline{AD}$, $\overline{DC}$, $\overline{CB}$

$\overline{EF}$ touches ⊙ at B

BA is at ⊥ with EF

centre of ⊙ is on BA [III.19]

BA is the diameter of ⊙ABCD

∡ADB = ⊥ [III.31]

∡BAD, ∡ABD = ⊥ [I.32]

∡ABF = ⊥

∡ABF = ∡BAD, ∡ABD

→ subtract ∡ABD

∡DBF = ∡BAD

in the alternate segment of ⊙

ABCD is a quadrilateral in a ⊙

opposite ∡ = 2⊥ [III.22]

∡BAD, ∡BCD = 2⊥

∡DBF, ∡DBE = 2⊥

∡DBF, ∡DBE = ∡BAD, ∡BCD

∡BAD = ∡DBF

↳ ∡DBE = ∡BCD

in the alternate segment DCB of ⊙

Q.E.D

Proposition 33

on a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle

given

$\overline{AB}$

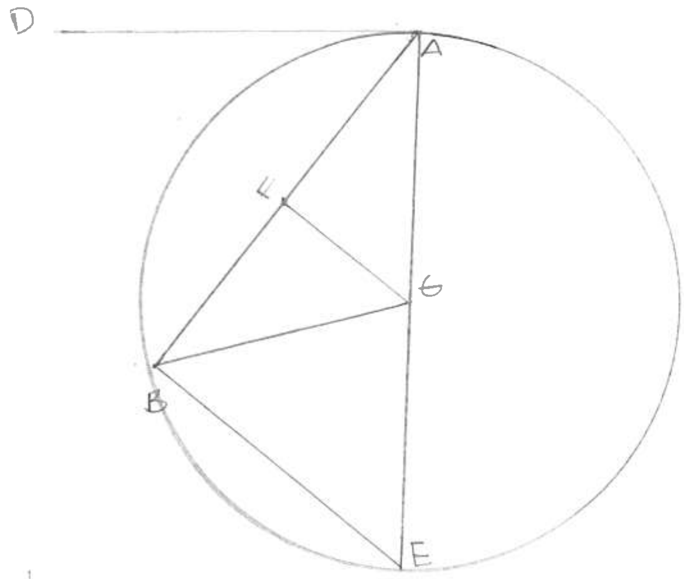
$\measuredangle C$

equired:

describe on AB

segment of a circle

dmitting an $\measuredangle = C$



Angle C is acute, right or obtuse

→ ACUTE

$\measuredangle BAD = C$

$\measuredangle BAD$ = acute $\measuredangle$

$\overline{AE}$ at ⊥ to $\overline{DA}$

$\overline{AB}$ bisected at F

$\overline{FG}$ at ⊥ to $\overline{AB}$

$\overline{GB}$

AF = FB

FG is common

AF, FG = FB, FG

$\measuredangle AFG = \measuredangle BFG$

base AG = base GB [I.4]

circle described with centre G & distance GA will pass through B

draw circle ABE

$\overline{EB}$

AD is drawn from A is at ⊥ with the diameter AE

AD touches the circle AB [III.16, Por]

AD touches ⊙ABE

from A, $\overline{AB}$ is drawn across ⊙ABE

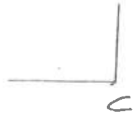
$\measuredangle DAB = \measuredangle AEB$ in the alternate segment [III.32]

$\measuredangle DAB = C$

$C = \measuredangle AEB$

Given a straight line AB, segment AEB of a circle has been described, admitting $\measuredangle AEB$ = to a given $\measuredangle$, $\measuredangle C$.

→RIGHT



$\angle C = \llcorner$

$\angle BAD = \angle C$

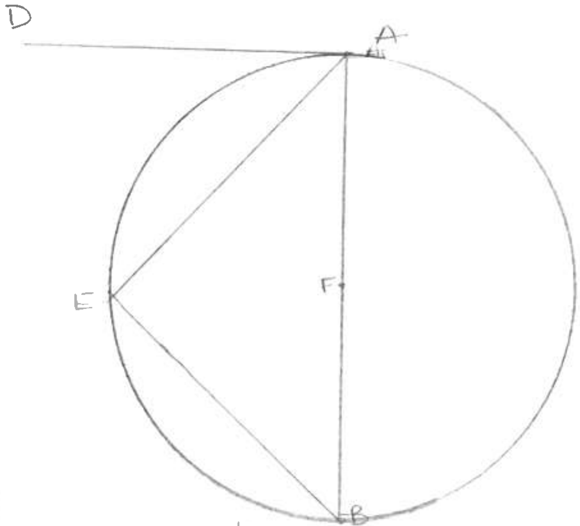
$\overline{AB}$ bisected at F

O AEB distance FB/FA

$\overline{AD}$ touches OABE
$\angle A = \llcorner$ [III.16, por]

$\angle BAD = \angle$ in segment AEB
$\angle AEB = \llcorner$ [III.31]
bargan ∡ ind

$\angle BAD = C$
$\angle AEB = C$



segment AEB of a O has been descuibed on AB admitting an $\angle = \angle C$

→OBTUSE

$\angle C$ = obtuse
$\angle BAD = \angle C$

AE at $\llcorner$ to AD
$\overline{AB}$ bisected at F
$\overline{FG}$ at $\llcorner$ to AB
$\overline{GB}$

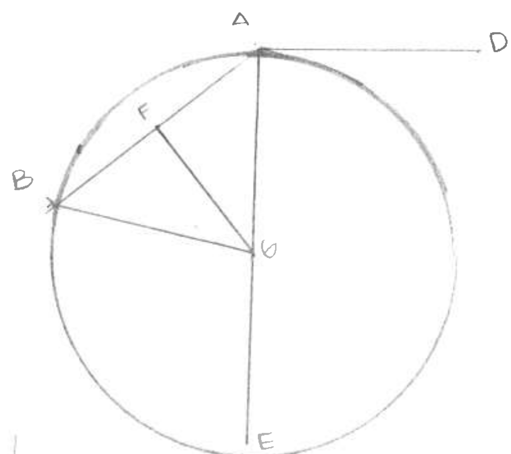
AF = FB
FG is common

AF, FG = FB, FG
$\angle AFG = \angle BFG$
base AG = base BG [I.4]

O AEB, centre G, distance GA will pass through B

AD is at $\llcorner$ to diameter AE from its extremity [III.16, por]
AD touches OAEB

AB drawn across from point of contact A. [III.32]
$\angle BAD = \angle$ constructed in alternate segment AHB of the O



$\angle BAD = \angle C$

$\angle AHB = \angle C$

on given line AB the segment AHB of a O has been descriped admitting an $\angle = \angle C$.

Q.E.F.

Proposition 34

from a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

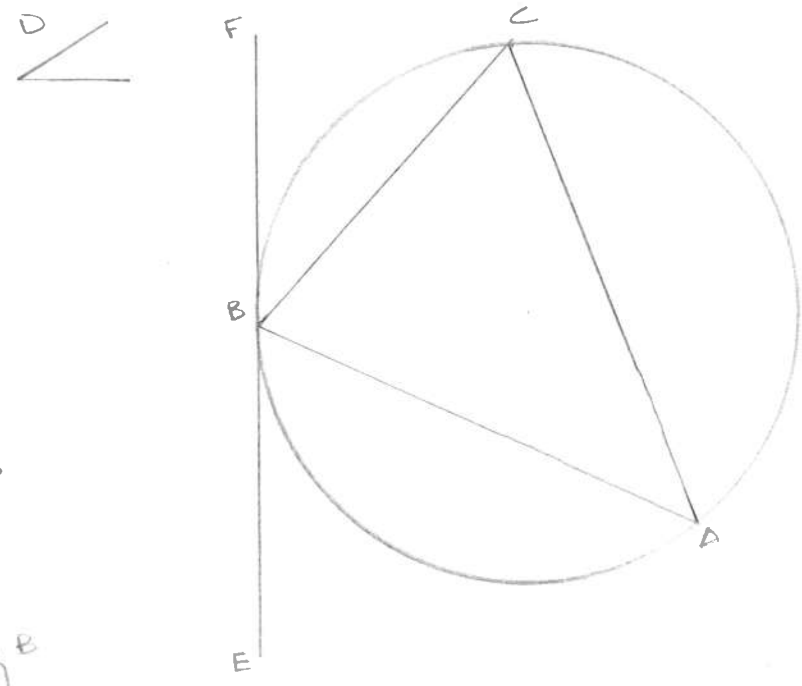
given

⊙ABC

∡D

required:

cut off from ⊙ABC a segment admitting ∡ = ∡D



EF touching ⊙ABC at B

∡FBC = ∡D [I.23]

EF touches ⊙ABC
BC drawn across from B (contact point)

∡FBC = ∡ in alternate segment BAC [III.32]

∡D = ∡FBC

∡ in segment BAC = ∡D

From a given circle ABC, the segment BAC has been cut off admitting ∡ = ∡D

Q.E.F.

Proposition 35

If in a given circle two straight lines cut one another, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of another.

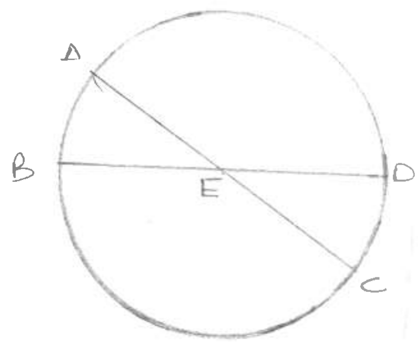
given

OABCD

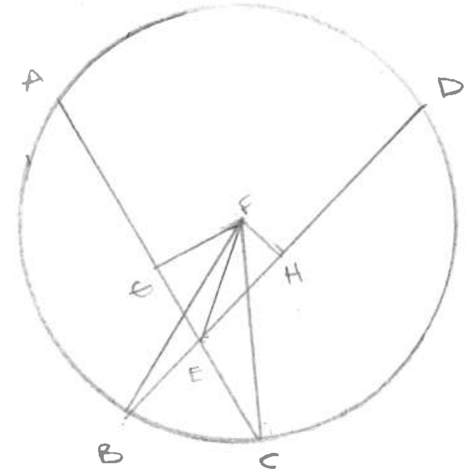
$\overline{AC}$, $\overline{BD}$ cut each other at E

required:

rectangle contained by AE, EC = rectangle contained by DE, EB.



a)



b)

→ (Figure a)

IF AC, BD are through the centre

E is centre of ABCD O

AE = EC = DE = EB

The rectangle contained by AE, EC = rectangle contained by DE, EB

→ (figure B)

AC, DE are <u>Not</u> through the centre.

F is centre OABCD

$\overline{FG}$, $\overline{FH}$ perpendicular to $\overline{AC}$, $\overline{DB}$

$\overline{FB}$, $\overline{FC}$, $\overline{FE}$

$\overline{GF}$ through the centre cuts $\overline{AC}$ not ⊥ through the centre at ⊥

GF bisects AC [III.3]

AG = GC

$\overline{AC}$ has been cut into equal parts at G and unequal parts at E

rectangle contained by AE, EC + square EG = square GC [II.5]

→ add square GF

rectangle AE, EC, squares EG, GF = squares GC, GF

square FE = squares GF, GE

square FC = squares GC, GF [I.47]

rectangle AE, EC, square FE = square FC

FC = FB

rectangle AE, EC, square FE = square FB

→ For the same reason,

rectangle DE, EB, square FE, = square FB

rectangle AE, EC, square FE = rectangle DE, EB, square FE

→ subtract FE

rectangle AE, EC = rectangle DE, EB

Q.E.D.

Proposition 36

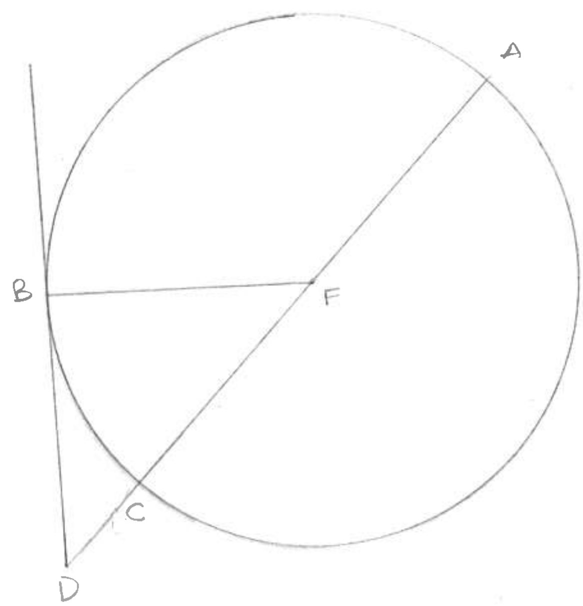
If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.

given:

·D outside ○ABC

$\overline{DCA}$ cut ○ABC

$\overline{DB}$ touch ○ABC



required

rectangle contained by AD, DC = square DB

→ DCA is through the centre

centre F

$\overline{FB}$

∡FBD = ⊾ [III.18]

$\overline{AC}$ bisected at F,
add $\overline{CD}$

rectangle AD, DC, square FC
= square FD [II.6]

FC = FB

rectangle AD, DC, square FB
= square FD

squares FB, BD = square FD [I.47]

rectangle AD, CD, square FB =
squares FB, BD

→ subtract FB

rectangle AC, CD = square BD #

→ DCA is not through the centre

E centre of ○ABC

$\overline{EF}$ perpendicular to AC

$\overline{EB}$, $\overline{EC}$, $\overline{ED}$

∡EBD = ⊾ [III.18]

EF cuts AC not through the centre at ⊾,
it bisects it [III.3]

AF = FC

AC bisected, $\overline{CD}$ added to it

rectangle AD, DC, square FC
= square FD [II.6]

→ add FE

rectangle AD, DC, square FC, FE
= square FD, FE

square EC = squares CF, FE
∡EFC = ⊾ [I.47]

square ED = squares DF, FE

rectangle AD, DC, square EC
= square ED

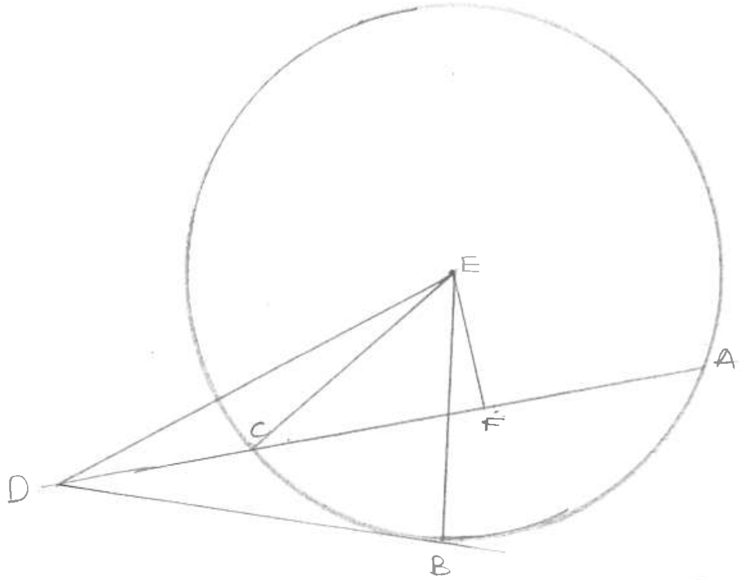
EC = EB

rectangle AD, DC, square EB
= square ED

squares EB, BD = square ED
∡EBD = ⊾ [I.47]

rectangle AD, DC, square EB
= square EB, BD

→ subtract EB

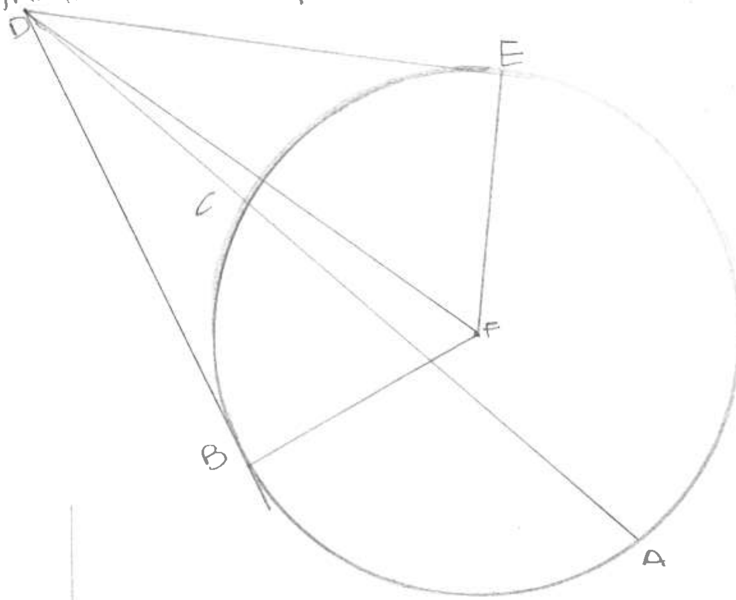


rectangle AD, DC = square BD

Q.E.D.

Proposition 37

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.



given:

•D outside ABC ○
DCA cut ○ABC
ectangle AD, DC =
square DB

equired

DB touches ○ABC

DE touching ○ABC
centre F
FE, FB, FD

∡FED = ⊾ [III.18]

DE touches ○ABC
DCA cuts it

ectangle AD, DC =
square DE [III.36]

ectangle AD, DC =
square DB

square DE = square DB
DE = DB

FE = FB
DE, EF = DB, BF
FD is common
∡DEF = ∡DBF [I.8]

∡DEF = ⊾

∡DBF = ⊾

FB produced is a diameter
a straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle. [III.16, POr]
DB touches the circle.

→ similarly it can be proved if the centre be on AC

QED